Thomaston Public Schools

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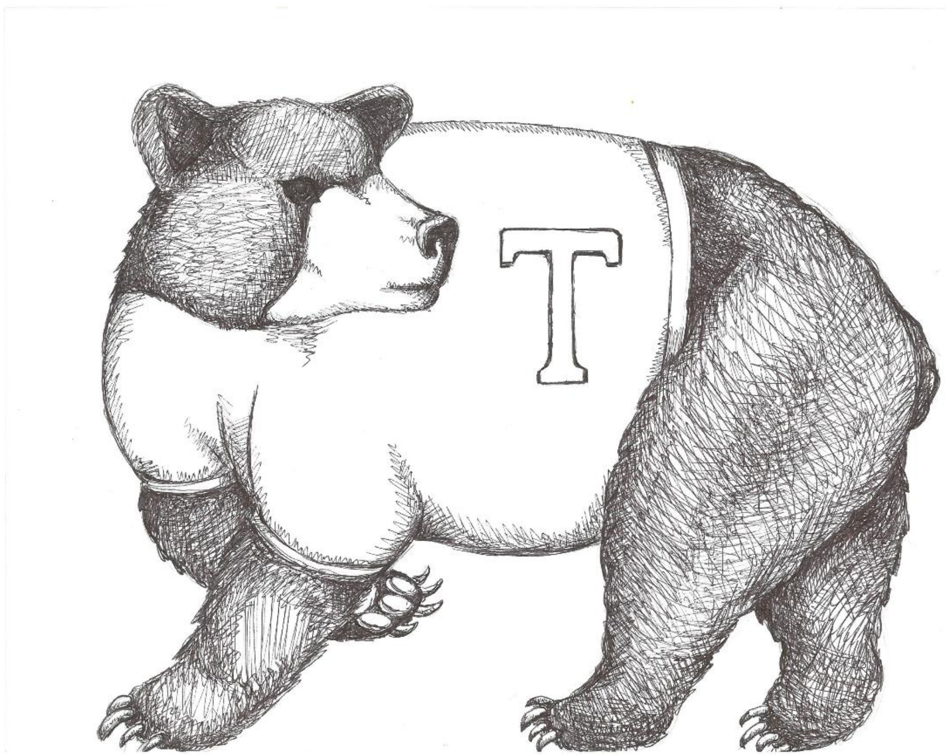

Thomaston Public Schools Curriculum
Thomaston High School
Mathematics: Precalculus 2015

Learn to Live ... Live to Learn

Acknowledgements

Curriculum Writer(s):

Mark Olsen

We acknowledge and celebrate the professionalism, expertise, and diverse perspectives of these teachers. Their contributions to this curriculum enrich the educational experiences of all Thomaston students.

______*Alisha DiCorpo*________________________________

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Precalculus

Precalculus

Board of Education Mission Statement:

IN A PARTNERSHIP OF FAMILY, SCHOOL AND COMMUNITY, OUR MISSION IS TO EDUCATE, CHALLENGE AND INSPIRE EACH INDIVIDUAL TO EXCEL AND BECOME A CONTRIBUTING MEMBER OF SOCIETY.

Departmental Philosophy:

The Mathematics Department strives to instill in each student a conceptual understanding of and procedural skill with the basic facts, principles and methods of mathematics. We want our students to develop an ability to explore, to make conjectures, to reason logically and to communicate mathematical ideas. We expect our students to learn to think critically and creatively in applying these ideas. We recognize that individual students learn in different ways and provide a variety of course paths and learning experiences from which students may choose. We emphasize the development of good writing skills and the appropriate use of technology throughout our curriculum. We hope that our students learn to appreciate mathematics as a useful discipline in describing and interpreting the world around us.

Main Resource used when writing this curriculum:

NYS COMMON CORE MATHEMATICS CURRICULUM A Story of Functions Curriculum. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. A Story of Functions: A Curriculum Overview for Grades 9-12 Date: 7/31/13 5 © 2013 Common Core, Inc. Some rights reserved. commoncore.org

Course Description:

Sequence of Precalculus Modules Aligned with the Standards

Unit 1: Complex Numbers and Transformations
Unit 2: Vectors and Matrices
Unit 3: Rational and Exponential Functions
Unit 4: Trigonometry
Unit 5: Probability and Statistics

Summary of Year

Extending their understanding of complex numbers to points in the complex plane, students come to understand that multiplying a given set of points by a complex number amounts to rotating and dilating those points in the complex plane about zero. Matrices are studied as tools for performing rotations and reflections of the coordinate plane, as well as for solving systems of linear equations. Inverse functions are explored as students study the relationship between exponential and logarithmic functions and restrict the domain of the trigonometric functions to allow for their inverses. The year concludes with a capstone module on modeling with probability and statistics. The Mathematical Practice Standards apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations.

Rationale for Module Sequence in Precalculus

In Algebra II, students extended their understanding of number to include complex numbers as they studied polynomials with complex zeros. In Module 1, students graph complex numbers in the complex plane and translate between rectangular and polar forms of complex numbers. In particular, through repeated reasoning, they come to realize that multiplying a given set of points by a complex number amounts to rotating and dilating those points in the complex plane around the point zero. Thinking of a complex number, a + bi, once again as a point (a,b) in the coordinate plane, students investigate how multiplying by a complex number can be thought of as a map from the coordinate plane to itself. That study, in turn, leads to matrix notation and a natural definition for multiplying a vector by a matrix. Thus, students discern structure in the operations with matrices and vectors by comparing them to arithmetic with complex numbers.

Students began the study of transformations in Grade 8, and precisely defined rigid motions in the plane in terms of angles, circles, perpendicular lines, parallel lines, and segments in Geometry. In this module, students precisely define rotations, reflections and dilations in the coordinate plane using 2×2 matrices. These well-defined definitions of transformations of the coordinate plane shed light on how geometry software and video games efficiently perform rigid motion calculations.

In the first module, students viewed matrices as tools for performing rotations and reflections of the coordinate plane. In Module 2, they move beyond this viewpoint to study matrices and vectors as objects in their own right. Students interpret the properties and operations of matrices to learn multiple ways to solve problems with them, including solving systems of linear equations. They construct viable arguments using matrices to once again derive equations for conic sections, this time by translating and rotating the locus of points into a “standard” position using matrix operations.

Students study rational and exponential functions in Module 3. They graph rational functions by extending what they learned about graphing polynomials functions. Students, through repeatedly exploiting the relationship between exponential and logarithmic functions, learn the meaning of inverse functions. Additionally, students learn to explicitly build composite functions to model relationships between two quantities. In particular, they analyze the composite of two functions in describing the relationship of three or more quantities in modeling activities in this module.

In Module 4, students visualize graphs of trigonometric functions with the aid of appropriate software and interpret how a family of graphs defined by varying a parameter in a given function changes based upon that parameter. They analyze symmetry and periodicity of trigonometric functions. They extend their knowledge of inverse functions to trigonometric functions by restricting domains to create the inverses, and apply inverse functions to solve trigonometric equations that arise in modeling contexts. Students also construct viable arguments to prove the Law of Sines, Law of Cosines, and the addition and subtraction formulas for the trigonometric functions.

This course concludes with Module 5, a capstone module on modeling with probability and statistics in which students consolidate their study of statistics as they analyze decisions and strategies using newly refined skills in calculating expected values.

Curriculum Map / Pacing Guide

Note: Adjustments should be made to accommodate testing schedules as they are made available. Pacing is based on the testing of the 2014-2015 school year.

	Grade 9 -- Algebra I	Grade 10 -- Geometry	Grade 11 -- Algebra II	Grade 12 -- Precalculus	
20 days	M1: Relationships Between Quantities and Reasoning with Equations and Their Graphs (40 days)	M1: Congruence, Proof, and Constructions (45 days)	M1: Polynomial, Rational, and Radical Relationships (45 days)	M1: Complex Numbers and Transformations (40 days)	20 days
20 days					20 days
20 days	M2: Descriptive Statistics (25 days)	M2: Similarity, Proof, and Trigonometry (45 days)	M2: Trigonometric Functions (20 days)	M2: Vectors and Matrices (40 days)	20 days
20 days	M3: Linear and Exponential Functions		M3: Functions (45 days)		20 days
	State Examinations	State Examinations	State Examinations	State Examinations	
20 days	(35 days)	M3: Extending to Three Dimensions (10 days)		M3: Rational and Exponential Functions (25 days)	20 days
20 days	M4: Polynomial and Quadratic Expressions, Equations and Functions (30 days)	M4: Connecting Algebra and Geometry through Coordinates (25 days)		M4: Trigonometry (20 days)	20 days
20 days			M4: Inferences and Conclusions from Data (40 days)	M5: Probability and Statistics (25 days)	20 days
20 days	M5: A Synthesis of Modeling with Equations and Functions (20 days)	M5: Circles with and Without Coordinates (25 days)			20 days
20 days	Review and Examinations	Review and Examinations	Review and Examinations	Review and Examinations	20 days

Key:	Number and Quantity and Modeling	Geometry and Modeling	Algebra and Modeling	Statistics and Probability and Modeling	Functions and Modeling

Complex Numbers and Transformations

Overview

Module 1 sets the stage for expanding students' understanding of transformations by first exploring the notion of linearity in an algebraic context ("Which familiar algebraic functions are linear?"). This quickly leads to a return to the study of complex numbers and a study of linear transformations in the complex plane. Thus, Module 1 builds on standards **N-CN.A.1** and **N-CN.A.2** introduced in the Algebra II course and standards **G-CO.A.2**, **G-CO.A.4**, and **G-CO.A.5** introduced in the Geometry course.

Topic A opens with a study of common misconceptions by asking questions such as "For which numbers □□ and □ does $(□ + □)^2 = □^2 + □^2$ happen to hold?" Some equations have only complex solutions, which launches a study of quotients of complex numbers and the use of conjugates to find moduli and quotients (**N-CN.A.3**). The topic ends by classifying real and complex functions that satisfy linearity conditions. (A function □ is linear if, and only if, there is a real or complex value □ such that □(□) = □ for all real or complex □.) Complex number multiplication is emphasized in the last lesson.

In Topic B, students develop an understanding that when complex numbers are considered points in the Cartesian plane, complex number multiplication has the geometric effect of a rotation followed by a dilation in the complex plan. This is a concept that has been developed since Algebra II and builds upon standards **N-CN.A.1** and **N-CN.A.2**, which, when introduced, were accompanied with the observation that multiplication by □□ has the geometric effect of rotating a given complex number 90° about the origin in a counterclockwise direction. The algebraic inverse of a complex number (its reciprocal) provides the inverse geometric operation. Analysis of the angle of rotation and the scale of the dilation brings a return to topics in trigonometry first introduced in Geometry (**G-SRT.C.6**, **G-SRT.C.7**, **G-SRT.C.8**) and expanded on in Algebra II (**F-TF.A.1**, **F-TF.A.2**, **F-TF.C.8**). It also reinforces the geometric interpretation of the modulus of a complex number and introduces the notion of the argument of a complex number.

The theme of Topic C is to highlight the effectiveness of changing notations and the power provided by certain notations such as matrices. By exploiting the connection to trigonometry, students see how much complex arithmetic is simplified. By regarding complex numbers as points in the Cartesian plane, students can begin to write analytic formulas for translations, rotations, and dilations in the plane and revisit the ideas of high school Geometry (**G-CO.A.2**, **G-CO.A.4**, **G-CO.A.5**) in this light. Taking this work one step further, students develop the 2×2 matrix notation for planar transformations represented by complex number arithmetic. This work sheds light on how geometry software and video games efficiently perform rigid motion calculations. Finally, the flexibility implied by 2×2 matrix notation allows students to study additional matrix transformations that do not necessarily arise from our original complex number thinking context.

In Topic C, the study of vectors and matrices is introduced through a coherent connection to transformations and complex numbers. Students learn to see matrices as representing transformations in the plane and develop understanding of multiplication of a matrix by a vector as a transformation acting on a point in the plane (**N-VM.C.11**, **N-VM.C.12**). While more formal study of multiplication of matrices will occur in Module 2, in Topic C, students are exposed to initial ideas of multiplying 2×2 matrices including a geometric interpretation of matrix invertibility and the meaning of the zero and identity matrices (**N-VM.C.8**, **N-VM.C.10**). **N-VM.C.8** is introduced in a strictly geometric context and is expanded upon more formally in

Module 2. **N-VM.C.8** will be assessed secondarily, in the context of other standards but not directly, in the Mid- and End-of-Module Assessments until Module 2.

The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic C.

Complex Numbers and Transformations

Unit 1

Subject: Mathematics

Grade/Course: Grade 12 / Precalculus

Pacing: 40 days

Unit of Study: Unit 1: Complex Numbers and Transformations

Priority Standards:

Perform arithmetic operations with complex numbers

N-CN.A.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

Represent complex numbers and their operations on the complex plane

N-CN.B.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.B.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(-1 + \sqrt{3}\, i)^3 = 8$ because $(-1 + \sqrt{3}\, i)$ has modulus 2 and argument 120°.

N-CN.B.6 (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Perform operations on matrices and use matrices in applications

N-VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N-VM.C.12 (+) Work with 2 x 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Foundational Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number □ such that $□^2 = -1$, and every complex number has the form □ + □ with □ and □ real.

N-CN.A.2 Use the relation $□^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. For example, rewrite $□^2 + 4$ as $(□ + 2□)(□ - 2□)$.

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $□(1 + □)^□$ as the product of □ and a factor not depending on □.

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. For example the expression $1.15^□$ can be rewritten as $(1.15^{1/12})^{12□} \approx 1.012^{12□}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

Create equations that describe numbers or relationships.

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm's law $V = IR$ to highlight resistance R.

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Experiment with transformations in the plane.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for □/3, □/4 and □/6, and use the unit circle to express the values of sine, cosine, and tangent for □ – □, □ + □, and 2□ – □ in terms of their values for □, where □ is any real number.

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity $\sin^2(□) + \cos^2(□) = 1$ and use it to find sin(□), cos(□), or tan(□) given sin(□), cos(□), or tan(□) and the quadrant of the angle.

Math Practice Standards:

MP.2 Reason abstractly and quantitatively.
Students come to recognize that multiplication by a complex number corresponds to the geometric action of a rotation and dilation from the origin in the complex plane. Students apply this knowledge to understand that multiplication by the reciprocal provides the inverse geometric operation to a rotation and dilation. Much of the module is dedicated to helping students quantify the rotations and dilations in increasingly abstract ways so they do not depend on the ability to visualize the transformation. That is, they reach a point where they do not need a specific geometric model in mind to think about a rotation or dilation. Instead, they can make generalizations about the rotation or dilation based on the problems they have previously solved.

MP.3 Construct viable arguments and critique the reasoning of others.
Throughout the module, students study examples of work by algebra students. This work includes a number of common mistakes that algebra students make, but it is up to the student to decide about the validity of the argument. Deciding on the validity of the argument focuses the students on justification and argumentation as they work to decide when purported algebraic identities do or do not hold. In cases where they decide that the given student work is incorrect, the students work to develop the correct general algebraic results and justify them by reflecting on what they perceived as incorrect about the original student solution.

MP.4 Model with mathematics.
As students work through the module, they become attuned to the geometric effect that occurs in the context of complex multiplication. However, initially it is unclear to them why multiplication by complex numbers entails specific geometric effects. In the module, the students create a model of computer animation in the plane. The focus of the mathematics in the computer animation is such that the students come to see rotating and translating as dependent on matrix operations and the addition of 2×1 vectors. Thus, their understanding becomes more formal with the notion of complex numbers.

“Unwrapped” Standards

N-CN.A.3 (+) Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

N-CN.B.4 (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N-CN.B.5 (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(-1 + \sqrt{3}\, i)^3 = 8$ because $(-1 + \sqrt{3}\, i)$ has modulus 2 and argument 120°.

N-CN.B.6 (+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.

Perform operations on matrices and use matrices in applications

N-VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

N-VM.C.12 (+) Work with 2 x 2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Concepts (What Students Need to Know)	**Skills (What Students Need to Be Able to Do)**
Conjugate of a complex number.	Find (L1)
Moduli and quotients of complex numbers.	Solve (L2) - using conjugates of complex numbers
Complex numbers and their operations.	Graph (L2) - on the complex plane in rectangular and polar form
Rectangular and polar forms of a complex number.	Compare (L2)
Operations on complex numbers.	Compute (L1) - using graphical representation
Distance between numbers in the complex plane.	Calculate (L1)
Midpoint between numbers in the complex plane.	Calculate (L1)
Zero and identity matrices.	Compare (L2) - to role of 0 and 1 in addition and multiplication
Determinant of a square matrix.	Understand (L2)
Vectors.	Transform (L4) - using matrices
Planes.	Transform (L4) - using 2 x 2 matrices.
Absolute value of the determinant of a matrix.	Interpret (L2) - in terms of area.

Essential Questions	**Big ideas**
How are algebraic expressions used to analyze and solve problems? What characteristics of problems would determine how to model the situation and develop a problem solving strategy? How does explaining the process help to understand a problem's solution better? When and why is it necessary to follow set rules/procedures/properties when manipulating numeric or algebraic expressions?	Algebraic expressions and equations generalize relationships from specific cases. Relationships can be described and generalizations made for mathematical situations that have numbers or objects that repeat in predictable ways. Multiple representations may be used to model given real world relationships. A problem solver understands what has been done, knows why the process was appropriate, and can support it with reasons and evidence. Rules of arithmetic and algebra can be used together with notions of equivalence to transform equations and inequalities.

Assessments		
Common Formative Pre-Assessments	Progress Monitoring Checks – "Dipsticks"	Common Formative Mid and or Post-Assessments Resources
Pretest vocabulary New terms (note: this list will be used for Unit 2 as well - it is a continuation of the same topics) Argument Bound Vector Complex Number Complex Plane Conjugate Determinant of ▯ × ▯ Matrix Determinant of ▯ × ▯ Matrix Directed Graph Directed Segment Free Vector Identity Matrix	Post-test vocabulary Opening Exercise - Give again and reflect on results from first administration prior to the unit. Exploratory Challenge Exit Ticket Student Conferences IXL Math	Type: Mid-Module Assessment Task Administered: After Topic B Format: Constructed response with rubric Standards Addressed: N-CN.A.3, N-CN.B.4, N-CN.B.5, N-CN.B.6 Type: End-of-Module Assessment Task Administered: After Topic C Format: Constructed response with rubric Standards Addresses: N-CN.B.4, N-CN.B.5, N-VM.C.8, N-VM.C.10, N-VM.C.11, N-VM.C.12

Imaginary Number Imaginary Part Imaginary Unit Incidence Matrix Inverse Matrix Linear Function Linear Transformation Linear Transformation Induced by Matrix ￿ Matrix Matrix Difference Matrix Product Matrix Scalar Multiplication Matrix Sum Modulus Network Diagram Opposite Vector Polar Form of a Complex Number Position Vector Real Coordinate Space Rectangular Form of a Complex Number Translation by a Vector in Real Coordinate Space Vector Addition Vector Subtraction Vector Magnitude Vector Scalar Multiplication Vector Representation of a Complex Number Zero Matrix Zero Vector Familiar Terms and Symbols Dilation Rectangular Form Rotation Translation Conduct opening exercise Use exit ticket as pre-assessment and post where applicable		

Performance Task
Performance tasks are to be created with teacher input throughout the year. A sample of a possible performance task detailed in Unit 4.
Engaging Learning Experiences
Engaging learning experiences are to be created with teacher input throughout the year. A sample of an engaging scenario is included in Unit 4.

Instructional Resources
Suggested Tools and Representations Graphing calculator ixl.com (Math) flippedmath.com (Precalculus) Geometer's Sketchpad software Wolfram Alpha software Lesson plans for all modules within the unit, that are compatible with this curriculum, can be found on EngageNY. The link below allows access to all lessons in Precalculus. (Just scroll down once you get there.) https://www.engageny.org/content/precalculus-and-advanced-topics (See Appendix A for an example.)

Instructional Strategies	Meeting the Needs of All Students

21st Century Skills Critical thinking and problem solving Collaboration and leadership Agility and Adaptability Effective oral and written communication Accessing and analyzing information **Marzano's Strategies** Identifying Similarities and Differences Reinforcing Effort and Providing Recognition Nonlinguistic Representations Homework and Practice Cooperative Learning Setting Objectives and Providing Feedback	The modules that make up Precalculus propose that the components of excellent math instruction do not change based on the audience. That said, there are specific resources included within this curriculum to highlight strategies that can provide critical access for all students. Researched-based Universal Design for Learning (UDL) has provided a structure for thinking about how to meet the needs of diverse learners. Broadly speaking, that structure asks teachers to consider multiple means of representation; multiple means of action and expression; and multiple means of engagement. Tables at the end of this section offer suggested scaffolds, utilizing this framework, for Students with Disabilities, Students Performing above Grade Level, and Students Performing below Grade Level. UDL offers ideal settings for multiple entry points for students and minimizes instructional barriers to learning. Teachers will note that many of the suggestions on a chart will be applicable to other students and overlapping populations. It is important to note that although the scaffolds/accommodations integrated into the course might change how a learner accesses information and demonstrates learning; they do not substantially alter the instructional level, content, or performance criteria. Rather, they provide students with choices in how they access content and demonstrate their knowledge and ability. **<u>Provide Multiple Means of Representation</u>** Teach from simple to complex, moving from concrete to abstract at the student's pace. Clarify, compare, and make connections to math words in discussion, particularly during and after practice. Partner key words with visuals and gestures. Connect language with concrete and pictorial experiences. Couple teacher-talk with "math-they-can-see," such as models. Let students use models and gestures to calculate and explain. For example, a student searching

	to define “multiplication” may model groups of 6 with drawings or concrete objects and write the number sentence to match. Teach students how to ask questions (such as “Do you agree?” and “Why do you think so?”) to extend “think-pair-share” conversations. Model and post conversation “starters,” such as: “I agree because…” “Can you explain how you solved it?” “I noticed that…” “Your solution is different from/ the same as mine because…” “My mistake was to…” Couple number sentences with models. Enlarge sprint print for visually impaired learners. Use student boards to work on one calculation at a time. Invest in or make math picture dictionaries or word walls. **<u>Provide Multiple Means of Action and Expression</u>** Provide a variety of ways to respond: oral; choral; student boards; concrete models, pictorial models; pair share; small group share. For example: Use student boards to adjust “partner share” for deaf and hard-of-hearing students. Partners can jot questions and answers to one another on slates. Use vibrations or visual signs (such as clap, rather than a snap or “show”) to elicit responses from deaf/hard of hearing students. Vary choral response with written response on student boards to ease linguistic barriers. Support oral or written response with sentence frames, such as “_______ is _____ hundreds, _____ tens, and _____ ones. Adjust oral fluency games by using student and teacher boards or hand signals, such as showing the sum with fingers. Use visual signals or vibrations to elicit responses. Adjust wait time for interpreters of deaf and hard-of-hearing students. Select numbers and tasks that are “just right” for learners.

	Model each step of the algorithm before students begin. Give students a chance to practice the next day's sprint beforehand. Give students a few extra minutes to process the information before giving the signal to respond. Assess by multiple means, including "show and tell" rather than written. Elaborate on the problem-solving process. Read word problems aloud. Post a visual display of the problem-solving process. Have students check off or highlight each step as they work. Talk through the problem-solving process step-by-step to demonstrate thinking process. Before students solve, ask questions for comprehension, such as, "What unit are we counting? What happened to the units in the story?" Teach students to use self-questioning techniques, such as, "Does my answer make sense?" Concentrate on goals for accomplishment within a time frame as opposed to a task frame. Extend time for task. Guide students to evaluate process and practice. Have students ask, "How did I improve? What did I do well?" Focus on students' mathematical reasoning (i.e., their ability to make comparisons, describe patterns, generalize, explain conclusions, specify claims, and use models), not their accuracy in language. **Provide Multiple Means of Engagement** Make eye-to-eye contact and keep teacher-talk clear and concise. Speak clearly when checking answers for sprints and problems. Check frequently for understanding (e.g., 'show'). Listen intently in order to uncover the math content in the students' speech. Use non-verbal signals, such as "thumbs-up." Assign a buddy or a group to clarify directions or process. Teach in small chunks so students get a lot of practice with one step at a time. Know, use, and make the most of Deaf culture and sign

	language. Use songs, rhymes, or rhythms to help students remember key concepts, such as “Add your ones up first/Make a bundle if you can!” Point to visuals and captions while speaking, using your hands to clearly indicate the image that corresponds to your words. Incorporate activity. Get students up and moving, coupling language with motion. Make the most of the fun exercises for activities like sprints and fluencies. Celebrate improvement. Intentionally highlight student math success frequently. Follow predictable routines to allow students to focus on content rather than behavior. Allow “everyday” and first language to express math understanding. Re-teach the same concept with a variety of fluency games. Allow students to lead group and pair-share activities. Provide learning aids, such as calculators and computers, to help students focus on conceptual understanding

New Vocabulary	**Students Achieving Below Standard**	**Students Achieving Above Standard**
New or Recently Introduced Terms (note: the same new vocabulary will be used for Unit 2) **Argument** - The argument of the complex number ⍰ is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray to the ray from the origin through the complex number ⍰ in the complex plane. **Bound Vector** - a directed line segment.	<u>**Provide Multiple Means of Representation**</u> Model problem-solving sets with drawings and graphic organizers (e.g., bar or tape diagram), giving many examples and visual displays. Guide students as they select and practice using their own graphic organizers and models to solve. Use direct instruction for vocabulary with visual or concrete	The following provides a bank of suggestions within the Universal Design for Learning framework for accommodating students who are above grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used. <u>**Provide Multiple Means of Representation**</u> Teach students how to ask questions

Complex Number - a number that can be represented by a point in the complex plane.

Complex Plane - a Cartesian plane such that the horizontal axis corresponding to points of the form (□, 0) is called the real axis, and a vertical axis corresponding to points of the form (0, □) is called the imaginary axis.

Conjugate - The conjugate of a complex number of the form □ + □□ is □ − □□.

Determinant of □ × □ Matrix

Determinant of □ × □ Matrix

Directed Graph - an ordered pair □ = (□, □) with □ a set whose elements are called vertices or nodes, and □ a set of ordered pairs of vertices, called arcs or directed edges.

Directed Segment - the line segment together with a direction given by connecting an initial point to a terminal point.

Free Vector - the equivalence class of all directed line segments (arrows) that are equivalent to each other by translation.

Identity Matrix

Imaginary Number - a complex number that can be expressed in the form □□ where □ is a real number.

Imaginary Part (See complex number.)

Imaginary Unit - the number corresponding to the point (0,1) in the complex plane.

Incidence Matrix - the □ × □ matrix such that the entry in row □ and column □ is the number of edges that

representations.

Use explicit directions with steps and procedures enumerated. Guide students through initial practice promoting gradual independence. "I do, we do, you do."

Use alternative methods of delivery of instruction such as recordings and videos that can be accessed independently or repeated if necessary.

Scaffold complex concepts and provide leveled problems for multiple entry points.

<u>Provide Multiple Means of Action and Expression</u>

First use manipulatives or real objects (such as dollar bills), then make transfer from concrete to pictorial to abstract.

Have students restate their learning for the day. Ask for a different representation in the restatement. 'Would you restate that answer in a different way or show me by using a diagram?'

Encourage students to explain their thinking and strategy for the solution.

Choose numbers and tasks that are "just right" for learners but teach the same concepts.

Adjust numbers in calculations to suit learner's levels. For example, change 429 divided by 2 to 400

(such as,"Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters," such as: "I agree because..." "Can you explain how you solved it?" "I noticed that...""Your solution is different from/ the same as mine because...""My mistake was to..."

Incorporate written reflection, evaluation, and synthesis.

Allow creativity in expression and modeling solutions.

<u>Provide Multiple Means of Action and Expression</u>

Encourage students to explain their reasoning both orally and in writing.

Extend exploration of math topics by means of challenging games, puzzles, and brain teasers.

Offer choices of independent or group assignments for early finishers.

Encourage students to notice and explore patterns and to identify rules and relationships in math. Have students share their observations in discussion and writing (e.g., journaling).

Foster their curiosity about numbers and mathematical ideas. Facilitate research and exploration through discussion, experiments, internet searches, trips, etc.
Have students compete in a secondary simultaneous competition,

start at node ⍰ and end at node ⍰. **Inverse Matrix** **Linear Function** - a polynomial function of degree one; that is, a function with real number domain and range that can be put into the form ⍰(⍰) = ⍰⍰ + ⍰ for real numbers ⍰ and ⍰. **Linear Transformation** **Linear Transformation Induced by Matrix ⍰** **Matrix** **Matrix Difference** **Matrix Product** **Matrix Scalar Multiplication** **Matrix Sum** **Modulus** - The modulus of a complex number ⍰ is the distance from the origin to the point corresponding to ⍰ in the complex plane. If ⍰ = ⍰ + ⍰⍰, then |⍰| = √⍰^{2} + ⍰^{2}. **Network Diagram** - a graphical representation of a directed graph where the ⍰ vertices are drawn as circles with each circle labeled by a number 1 through ⍰, and the directed edges are drawn as segments or arcs with arrow pointing from the tail vertex to the head vertex. **Opposite Vector** **Polar Form of a Complex Number** - The polar form of a complex number ⍰ is ⍰(cos(⍰) + ⍰ sin(⍰)) where ⍰ = |⍰| and ⍰ = arg(⍰). **Position Vector** - a free vector that is represented by the directed line	divided by 2 or 4 divided by 2. **<u>Provide Multiple Means of Engagement</u>** Clearly model steps, procedures, and questions to ask when solving. Cultivate peer-assisted learning interventions for instruction (e.g., dictation) and practice, particularly for computation work (e.g., peer modeling). Have students work together to solve and then check their solutions. Teach students to ask themselves questions as they solve: Do I know the meaning of all the words in this problem?; What is being asked?; Do I have all of the information I need?; What do I do first?; What is the order to solve this problem? What calculations do I need to make? Practice routine to ensure smooth transitions. Set goals with students regarding the type of math work students should complete in 60 seconds. Set goals with the students regarding next steps and what to focus on next. Reinforce foundational standards (listed after priority standards) for the unit.	such as skip-counting by 75s, while peers are completing the sprint. Let students choose their mode of response: written, oral, concrete, pictorial, or abstract. Increase the pace. Offer two word problems to solve, rather than one. Adjust difficulty level by increasing the number of steps (e.g., change a one-step problem to a two-step problem). Adjust difficulty level by enhancing the operation (e.g., addition to multiplication), increasing numbers to millions, or decreasing numbers to decimals/fractions. Let students write word problems to show mastery and/or extension of the content. **<u>Provide Multiple Means of Engagement</u>** Push student comprehension into higher levels of Bloom's Taxonomy with questions such as: "What would happen if...?""Can you propose an alternative...?""How would you evaluate...?""What choice would you have made...?" Ask "Why?" and "What if?" questions. Celebrate improvement in completion time (e.g., Sprint A completed in 45 seconds and Sprint B completed

segment from the origin to the point. **Real Coordinate Space** - One-dimensional real coordinate space is called a number line and the two-dimensional real coordinate space is called the Cartesian plane. **Rectangular Form of a Complex Number** - The rectangular form of a complex number □ is □ + □□ where □ corresponds to the point (□, □) in the complex plane, and □ is the imaginary unit. **Translation by a Vector in Real Coordinate Space** **Vector Addition** **Vector Subtraction** **Vector Magnitude** - the length of any directed line segment that represents the vector **Vector Scalar Multiplication** **Vector Representation of a Complex Number** - the position vector associated to the point in the complex plane. **Zero Matrix** - the □ × □ matrix in which all entries are equal to zero. **Zero Vector** - the vector in which each component is equal to zero. Familiar Terms and Symbols Dilation Rectangular Form Rotation Translation		in 30 seconds). Make the most of the fun exercises for practicing skip-counting. Accept and elicit student ideas and suggestions for ways to extend games. Cultivate student persistence in problem-solving and do not neglect their need for guidance and support.

Vectors and Matrices

Overview

In Module 1 students learned that throughout the 1800s, mathematicians encountered a number of disparate situations that seemed to call for displaying information via tables and performing arithmetic operations on those tables. One such context arose in Module 1, where students saw the utility of representing linear transformations in the two-dimensional coordinate plane via matrices. Students viewed matrices as representing transformations in the plane and developed an understanding of multiplication of a matrix by a vector as a transformation acting on a point in the plane. This module starts with a second context for matrix representation, networks.

In Topic A, students look at incidence relationships in networks and encode information about them via highdimensional matrices **(N-VM.C.6)**. Questions on counting routes, the results of combining networks, payoffs, and other applications, provide context and use for matrix manipulations: matrix addition and subtraction, matrix product, and multiplication of matrices by scalars **(N-VM.C.7, N-VM.C.8)**.

The question naturally arises as to whether there is a geometric context for higher-dimensional matrices as there is for 2×2 matrices. Topic B explores this question, extending the concept of a linear transformation from Module 1 to linear transformations in three- (and higher-) dimensional space. The geometric effect of matrix operations—matrix product, matrix sum, and scalar multiplication—are examined, and students come to see, geometrically, that matrix multiplication for square matrices is not a commutative operation, but that it still satisfies the associative and distributive properties **(N-VM.C.9)**. The geometric and arithmetic roles of the zero matrix and identity matrix are discussed, and students see that a multiplicative inverse to a square matrix exists precisely when the determinant of the matrix (given by the area of the image of the unit square in two-dimensional space, or the volume of the image of the unit cube in three-dimensional space) is nonzero **(N-VM.C.10)**. This work is phrased in terms of matrix operations on vectors, seen as matrices with one column **(N-VM.C.11)**.

Topic C provides a third context for the appearance of matrices via the study of systems of linear equations. Students see that a system of linear equations can be represented as a single matrix equation in a vector variable **(A-REI.C.8)**, and that one can solve the system with the aid of the multiplicative inverse to a matrix if it exists **(A-REI.C.9**).

Topic D opens with a formal definition of a vector (the motivation and context for it is well in place at this point) and the arithmetical work for vector addition, subtraction, scalar multiplication, and vector magnitude is explored along with the geometrical frameworks for these operations **(N-VM.A.1, N-VM.A.2, N-VM.B.4, N-VM.B.5)**. Students also solve problems involving velocity and other quantities that can be represented by vectors **(N-VM.A.3)**. Parametric equations are introduced in Topic D allowing students to connect their prior work with functions to vectors.

The module ends with Topic E where students apply their knowledge developed in this module to understand how first-person video games use matrix operations to project three-dimensional objects onto two-dimensional screens and animate those images to give the illusion of motion **(N-VM.C.8, N-VM.C.9, N-VM.C.10, N-VM.C.11).**

Vectors and Matrices

Unit 2

Subject: Mathematics

Grade/Course: Grade 12 / Precalculus

Pacing: 40 days

Unit of Study: Unit 2: Vectors and Matrices

Priority Standards:

Represent and model with vector quantities

N-VM.A.1 (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, |v|, ||v||, v).

N-VM.A.2 (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM.A.3 (+) Solve problems involving velocity and other quantities that can be represented by vectors.

Perform operations on vectors

N-VM.B.4 (+) Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction v – w as v + (–w), where –w is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM.B.5 (+) Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as c(vx, vy) = (cvx, cvy).

b. Compute the magnitude of a scalar multiple cv using ||cv|| = |c|v. Compute the direction of cv knowing that when |c|v ≠ 0, the direction of cv is either along v for (c > 0) or against v (for c < 0).

Perform operations on matrices and use matrices in applications.

N-VM.C.6 (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM.C.7 (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

N-VM.C.8 (+) Add, subtract, and multiply matrices of appropriate dimensions.

N-VM.C.9 (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

N-VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.

N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

Solve systems of equations

A-REI.C.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable.

A-REI.C.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 x 3 or greater).

Foundational Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal, and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

Create equations that describe numbers or relationships.

A-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm's law $V = IR$ to highlight resistance R.

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Extend the domain of trigonometric functions using the unit circle.

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for ⍰/3, ⍰/4 and ⍰/6, and use the unit circle to express the values of sine, cosine, and tangent for ⍰ – ⍰, ⍰ + ⍰, and 2⍰ – ⍰ in terms of their values for ⍰, where ⍰ is any real number.

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity sin^2 (⍰) + cos^2 (⍰) = 1 and use it to find sin(⍰), cos(⍰), or tan(⍰) given sin(⍰), cos(⍰), or tan(⍰) and the quadrant of the angle.

Experiment with transformations in the plane.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not.

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Translate between the geometric description and the equation for a conic section.

G-GPE.A.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

G-GPE.A.2 Derive the equation of a parabola given a focus and directrix.

Math Practice Standards:

MP.2 Reason abstractly and quantitatively. Students recognize matrices and justify the transformations that they represent. Students use 3 × 3 matrices to solve systems of equations and continue to calculate the determinant of matrices. Students also represent complex numbers as vectors and determine magnitude and direction. Students reason to determine the effect of scalar multiplication and the result of a zero vector.

MP.4 Model with mathematics. Students initially study matrix multiplication as a tool for modeling networks and create a model of a bus route. Later, students look at matrix transformations and their role in developing video games and create their own video game. The focus of the mathematics in the computer animation is such that the students come to see rotating and translating as dependent on matrix operations and the addition vectors.

MP.5 Use appropriate tools strategically. As students study 3 × 3 matrices, they begin to view matrices as a tool that can solve problems including networks, payoffs, velocity, and force. Students use calculators and computer software to solve systems of three equations and three unknowns using matrices. Computer software is also used to help students visualize three-dimensional changes on a two-dimensional screen and in the creation of their video games.

"Unwrapped" Standards

N-VM.A.1 (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., v, |v|, ||v||, v).

N-VM.A.2 (+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM.A.3 (+) Solve problems involving velocity and other quantities that can be represented by vectors.

N-VM.B.4 (+) Add and subtract vectors.

a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.

b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.

c. Understand vector subtraction v – w as v + (–w), where –w is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM.B.5 (+) Multiply a vector by a scalar.

a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as c(vx, vy) = (cvx, cvy).

b. Compute the magnitude of a scalar multiple cv using ||cv|| = |c|v. Compute the direction of cv knowing that when |c|v ≠ 0, the direction of cv is either along v for (c > 0) or against v (for c < 0).

N-VM.C.6 (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM.C.7 (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.

N-VM.C.8 (+) Add, subtract, and multiply matrices of appropriate dimensions. N-VM.C.9 (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties. N-VM.C.10 (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse. N-VM.C.11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors. A-REI.C.8 (+) Represent a system of linear equations as a single matrix equation in a vector variable. A-REI.C.9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 x 3 or greater).	
Concepts (What Students Need to Know)	**Skills (What Students Need to Be Able to Do)**
Vector quantities.	Recognize (L1) Represent (L2)
Components of a vector.	Find (L1) - by subtraction
Problems that can be represented by vectors.	Solve (L2)
Vector Addition.	Compute (L1) - using multiple strategies Graph (L2)
Scalar multiplication of Vector.	Compute (L1)- using multiple strategies Graph (L2)
Matrices.	Represent (L2) - data Manipulate (L2) - data Add (L1) Subtract (L1) Multiply (L1)
Properties of matrix operations.	Understand (L2) Explain (L2) Use (L1)
Zero and identity matrices.	Compare (L2) - to role of 0 and 1 in addition and multiplication
Determinant of a square matrix.	Understand (L2)
Vectors.	Transform (L4) - using matrices
Planes.	Transform (L4) - using 2 x 2 matrices

Inverse of a matrix. System of linear equations.	Find (L1) Represent (L2) - as a single matrix equation Solve (L2) - using inverse of a matrix

Essential Questions	**Big ideas**
How are algebraic expressions used to analyze and solve problems? What characteristics of problems would determine how to model the situation and develop a problem solving strategy? How does explaining the process help to understand a problem's solution better? When and why is it necessary to follow set rules/procedures/properties when manipulating numeric or algebraic expressions?	Algebraic expressions and equations generalize relationships from specific cases. Relationships can be described and generalizations made for mathematical situations that have numbers or objects that repeat in predictable ways. Multiple representations may be used to model given real world relationships. A problem solver understands what has been done, knows why the process was appropriate, and can support it with reasons and evidence. Rules of arithmetic and algebra can be used together with notions of equivalence to transform equations and inequalities.

Assessments		
Common Formative Pre-Assessments	Progress Monitoring Checks – "Dipsticks"	Common Formative Mid and or Post-Assessments Resources
Pretest vocabulary	Post-test vocabulary	Type: Mid-Module Assessment

(note: this is the same list as Unit 1) Argument Bound Vector Complex Number Complex Plane Conjugate Determinant of □ × □ Matrix Determinant of □ × □ Matrix Directed Graph Directed Segment Free Vector Identity Matrix Imaginary Number Imaginary Part Imaginary Unit Incidence Matrix Inverse Matrix Linear Function Linear Transformation Linear Transformation Induced by Matrix □ Matrix Matrix Difference Matrix Product Matrix Scalar Multiplication Matrix Sum Modulus Network Diagram Opposite Vector Polar Form of a Complex Number Position Vector Real Coordinate Space Rectangular Form of a Complex Number Translation by a Vector in Real Coordinate Space Vector Addition Vector Subtraction Vector Magnitude Vector Scalar Multiplication Vector Representation of a Complex Number Zero Matrix Zero Vector Familiar Terms and Symbols Dilation Rectangular Form	Opening Exercise - Give again and reflect on results from first administration prior to the unit. Exploratory Challenge Exit Ticket Student Conferences IXL Math	Task Administered: After Topic B Format: Constructed response with rubric Standards Addressed: N-VM.C.6, N-VM.C.7, N-VM.C.8, N-VM.C.9, N-VM.C.10, N-VM.C.11 Type: End-of-Module Assessment Task Administered: After Topic C Format: Constructed response with rubric Standards Addresses: N-VM.A.1, N-VM.A.2, N-VM.A.3, N-VM.B.4, N-VM.B.5, N-VM.C.8, N-VM.C.11, A-REI.C.8, A-REI.C.9

Rotation Translation Conduct opening exercise Use exit ticket as pre-assessment and post where applicable		

Performance Task
Performance tasks are to be created with teacher input throughout the year. A sample of a possible performance task detailed in Unit 4.

Engaging Learning Experiences
Engaging learning experiences are to be created with teacher input throughout the year. A sample of an engaging scenario is included in Unit 4.

Instructional Resources
Suggested Tools and Representations Graphing calculator ixl.com (Math) flippedmath.com (Precalculus) Geometer's Sketchpad software Wolfram Alpha software ALICE 3.1

Geogebra Software

Lesson plans for all modules within the unit, that are compatible with this curriculum, can be found on EngageNY. The link below allows access to all lessons in Precalculus. (Just scroll down once you get there.) https://www.engageny.org/content/precalculus-and-advanced-topics (See Appendix A for an example.)

Instructional Strategies	Meeting the Needs of All Students
21st Century Skills Critical thinking and problem solving Collaboration and leadership Agility and Adaptability Effective oral and written communication Accessing and analyzing information **Marzano's Strategies** Identifying Similarities and Differences Reinforcing Effort and Providing Recognition Nonlinguistic Representations Homework and Practice Cooperative Learning Setting Objectives and Providing Feedback	The modules that make up Precalculus propose that the components of excellent math instruction do not change based on the audience. That said, there are specific resources included within this curriculum to highlight strategies that can provide critical access for all students. Researched-based Universal Design for Learning (UDL) has provided a structure for thinking about how to meet the needs of diverse learners. Broadly speaking, that structure asks teachers to consider multiple means of representation; multiple means of action and expression; and multiple means of engagement. Tables at the end of this section offer suggested scaffolds, utilizing this framework, for Students with Disabilities, Students Performing above Grade Level, and Students Performing below Grade Level. UDL offers ideal settings for multiple entry points for students and minimizes instructional barriers to learning. Teachers will note that many of the suggestions on a chart will be applicable to other students and overlapping populations. It is important to note that although the scaffolds/accommodations integrated into the course might change how a learner accesses information and demonstrates learning; they do not substantially alter the instructional level, content, or performance criteria. Rather, they provide students with choices in how they access content and demonstrate their knowledge and ability. **Provide Multiple Means of Representation** Teach from simple to complex, moving from concrete to abstract at the student's pace. Clarify, compare, and make connections to math words in discussion, particularly during and after practice. Partner key words with visuals and gestures. Connect language with concrete and pictorial experiences.

	Couple teacher-talk with "math-they-can-see," such as models. Let students use models and gestures to calculate and explain. For example, a student searching to define "multiplication" may model groups of 6 with drawings or concrete objects and write the number sentence to match. Teach students how to ask questions (such as "Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters," such as: "I agree because…" "Can you explain how you solved it?" "I noticed that…" "Your solution is different from/ the same as mine because…" "My mistake was to…" Couple number sentences with models. Enlarge sprint print for visually impaired learners. Use student boards to work on one calculation at a time. Invest in or make math picture dictionaries or word walls. **<u>Provide Multiple Means of Action and Expression</u>** Provide a variety of ways to respond: oral; choral; student boards; concrete models, pictorial models; pair share; small group share. For example: Use student boards to adjust "partner share" for deaf and hard-of-hearing students. Partners can jot questions and answers to one another on slates. Use vibrations or visual signs (such as clap, rather than a snap or "show") to elicit responses from deaf/hard of hearing students. Vary choral response with written response on student boards to ease linguistic barriers. Support oral or written response with sentence frames, such as "______ is ____ hundreds, ____ tens, and ____ ones. Adjust oral fluency games by using student and teacher boards or hand signals, such as showing the sum with fingers. Use visual signals or vibrations to elicit responses. Adjust wait time for interpreters of deaf and hard-of-hearing students. Select numbers and tasks that are "just right" for learners. Model each step of the algorithm before students begin. Give students a chance to practice the next day's sprint beforehand. Give students a few extra minutes to process the

	information before giving the signal to respond. Assess by multiple means, including “show and tell” rather than written. Elaborate on the problem-solving process. Read word problems aloud. Post a visual display of the problem-solving process. Have students check off or highlight each step as they work. Talk through the problem-solving process step-by-step to demonstrate thinking process. Before students solve, ask questions for comprehension, such as, “What unit are we counting? What happened to the units in the story?” Teach students to use self-questioning techniques, such as, “Does my answer make sense?” Concentrate on goals for accomplishment within a time frame as opposed to a task frame. Extend time for task. Guide students to evaluate process and practice. Have students ask, “How did I improve? What did I do well?” Focus on students’ mathematical reasoning (i.e., their ability to make comparisons, describe patterns, generalize, explain conclusions, specify claims, and use models), not their accuracy in language. **<u>Provide Multiple Means of Engagement</u>** Make eye-to-eye contact and keep teacher-talk clear and concise. Speak clearly when checking answers for sprints and problems. Check frequently for understanding (e.g., ‘show’). Listen intently in order to uncover the math content in the students’ speech. Use non-verbal signals, such as “thumbs-up.” Assign a buddy or a group to clarify directions or process. Teach in small chunks so students get a lot of practice with one step at a time. Know, use, and make the most of Deaf culture and sign language. Use songs, rhymes, or rhythms to help students remember key concepts, such as “Add your ones up first/Make a bundle if you can!” Point to visuals and captions while speaking, using your hands to clearly indicate the image that corresponds to your words. Incorporate activity. Get students up and moving, coupling language with motion. Make the most of the fun exercises for activities like sprints and fluencies. Celebrate improvement. Intentionally highlight student

	math success frequently. Follow predictable routines to allow students to focus on content rather than behavior. Allow “everyday” and first language to express math understanding. Re-teach the same concept with a variety of fluency games. Allow students to lead group and pair-share activities. Provide learning aids, such as calculators and computers, to help students focus on conceptual understanding

New Vocabulary	Students Achieving Below Standard	Students Achieving Above Standard
New or Recently Introduced Terms (note: this is the same new vocabulary used in Unit 1) **Argument** - The argument of the complex number □ is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray to the ray from the origin through the complex number □ in the complex plane. **Bound Vector** - a directed line segment. **Complex Number** - a number that can be represented by a point in the complex plane. **Complex Plane** - a Cartesian plane such that the horizontal axis corresponding to points of the form (□, 0) is called the real axis, and a vertical axis corresponding to points of the form (0, □) is called the imaginary axis. **Conjugate** - The conjugate of a complex number of the form □ + □□ is □ − □□. **Determinant of □ × □ Matrix**	**<u>Provide Multiple Means of Representation</u>** Model problem-solving sets with drawings and graphic organizers (e.g., bar or tape diagram), giving many examples and visual displays. Guide students as they select and practice using their own graphic organizers and models to solve. Use direct instruction for vocabulary with visual or concrete representations. Use explicit directions with steps and procedures enumerated. Guide students through initial practice promoting gradual independence. “I do, we do, you do.” Use alternative methods of delivery of instruction such as recordings and videos that can be accessed independently or repeated if necessary. Scaffold complex concepts and provide leveled problems for multiple entry points.	The following provides a bank of suggestions within the Universal Design for Learning framework for accommodating students who are above grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used. **<u>Provide Multiple Means of Representation</u>** Teach students how to ask questions (such as,“Do you agree?” and “Why do you think so?”) to extend “think-pair-share” conversations. Model and post conversation “starters,” such as: “I agree because...” “Can you explain how you solved it?” “I noticed that...”“Your solution is different from/ the same as mine because...”“My mistake was to...” Incorporate written reflection, evaluation, and synthesis. Allow creativity in expression and modeling solutions.

Determinant of □ × □ Matrix **Directed Graph** - an ordered pair □ = (□, □) with □ a set whose elements are called vertices or nodes, and □ a set of ordered pairs of vertices, called arcs or directed edges. **Directed Segment** - the line segment together with a direction given by connecting an initial point to a terminal point. **Free Vector** - the equivalence class of all directed line segments (arrows) that are equivalent to each other by translation. **Identity Matrix** **Imaginary Number** - a complex number that can be expressed in the form □□ where □ is a real number. **Imaginary Part** (See complex number.) **Imaginary Unit** - the number corresponding to the point (0,1) in the complex plane. **Incidence Matrix** - the □ × □ matrix such that the entry in row □ and column □ is the number of edges that start at node □ and end at node □. **Inverse Matrix** **Linear Function** - a polynomial function of degree one; that is, a function with real number domain and range that can be put into the form □(□) = □□ + □ for real numbers □ and □. **Linear Transformation** **Linear Transformation Induced by Matrix □** **Matrix**	 **<u>Provide Multiple Means of Action and Expression</u>** First use manipulatives or real objects (such as dollar bills), then make transfer from concrete to pictorial to abstract. Have students restate their learning for the day. Ask for a different representation in the restatement. 'Would you restate that answer in a different way or show me by using a diagram?' Encourage students to explain their thinking and strategy for the solution. Choose numbers and tasks that are "just right" for learners but teach the same concepts. Adjust numbers in calculations to suit learner's levels. For example, change 429 divided by 2 to 400 divided by 2 or 4 divided by 2. **<u>Provide Multiple Means of Engagement</u>** Clearly model steps, procedures, and questions to ask when solving. Cultivate peer-assisted learning interventions for instruction (e.g., dictation) and practice, particularly for computation work (e.g., peer modeling). Have students work together to solve and then check their solutions. Teach students to ask themselves	**<u>Provide Multiple Means of Action and Expression</u>** Encourage students to explain their reasoning both orally and in writing. Extend exploration of math topics by means of challenging games, puzzles, and brain teasers. Offer choices of independent or group assignments for early finishers. Encourage students to notice and explore patterns and to identify rules and relationships in math. Have students share their observations in discussion and writing (e.g., journaling). Foster their curiosity about numbers and mathematical ideas. Facilitate research and exploration through discussion, experiments, internet searches, trips, etc. Have students compete in a secondary simultaneous competition, such as skip-counting by 75s, while peers are completing the sprint. Let students choose their mode of response: written, oral, concrete, pictorial, or abstract. Increase the pace. Offer two word problems to solve, rather than one. Adjust difficulty level by increasing the number of steps (e.g., change a one-step problem to a two-step problem).

Matrix Difference **Matrix Product** **Matrix Scalar Multiplication** **Matrix Sum** **Modulus** - The modulus of a complex number ▯ is the distance from the origin to the point corresponding to ▯ in the complex plane. If ▯ = ▯ + ▯▯, then |▯| = √▯^{2} + ▯^{2}. **Network Diagram** - a graphical representation of a directed graph where the ▯ vertices are drawn as circles with each circle labeled by a number 1 through ▯, and the directed edges are drawn as segments or arcs with arrow pointing from the tail vertex to the head vertex. **Opposite Vector** **Polar Form of a Complex Number** - The polar form of a complex number ▯ is ▯(cos(▯) + ▯ sin(▯)) where ▯ = |▯| and ▯ = arg(▯). **Position Vector** - a free vector that is represented by the directed line segment from the origin to the point. **Real Coordinate Space** - One-dimensional real coordinate space is called a number line and the two-dimensional real coordinate space is called the Cartesian plane. **Rectangular Form of a Complex Number** - The rectangular form of a complex number ▯ is ▯ + ▯▯ where ▯ corresponds to the point (▯, ▯) in the complex plane, and ▯ is the imaginary unit. **Translation by a Vector in Real Coordinate Space**	questions as they solve: Do I know the meaning of all the words in this problem?; What is being asked?; Do I have all of the information I need?; What do I do first?; What is the order to solve this problem? What calculations do I need to make? Practice routine to ensure smooth transitions. Set goals with students regarding the type of math work students should complete in 60 seconds. Set goals with the students regarding next steps and what to focus on next. Reinforce foundational standards (listed after priority standards) for the unit.	Adjust difficulty level by enhancing the operation (e. g., addition to multiplication), increasing numbers to millions, or decreasing numbers to decimals/fractions. Let students write word problems to show mastery and/or extension of the content. **<u>Provide Multiple Means of Engagement</u>** Push student comprehension into higher levels of Bloom's Taxonomy with questions such as: "What would happen if...?""Can you propose an alternative...?""How would you evaluate...?""What choice would you have made...?" Ask "Why?" and "What if?" questions. Celebrate improvement in completion time (e.g., Sprint A completed in 45 seconds and Sprint B completed in 30 seconds). Make the most of the fun exercises for practicing skip-counting. Accept and elicit student ideas and suggestions for ways to extend games. Cultivate student persistence in problem-solving and do not neglect their need for guidance and support.

Vector Addition **Vector Subtraction** **Vector Magnitude** - the length of any directed line segment that represents the vector **Vector Scalar Multiplication** **Vector Representation of a Complex Number** - the position vector associated to the point in the complex plane. **Zero Matrix** - the ▯ × ▯ matrix in which all entries are equal to zero. **Zero Vector** - the vector in which each component is equal to zero. Familiar Terms and Symbols Dilation Rectangular Form Rotation Translation		

Rational and Exponential Functions

Overview

Students encountered the fundamental theorem of algebra, that every polynomial function has at least one zero in the realm of the complex numbers **(N-CN.C.9)**, in Algebra II, Module 1. Topic A of this module brings students back to the study of complex roots of polynomial functions. Students first briefly review quadratic and cubic functions and then extend familiar polynomial identities to both complex numbers **(N-CN.C.8)** and to general polynomial functions. Students use polynomial identities to find square roots of complex numbers. The binomial theorem and its relationship to Pascal's Triangle are explored using roots of unity **(A-APR.C.5).** Topic A concludes with students' use of Cavalieri's principle to derive formulas **(G-GPE.A.3)** for the volume of the sphere and other geometric solids **(G-GMD.A.2)**.

In Topic B, students explore composition of functions in depth **(F-BF.A.1c)** and notice that a composition of a polynomial function with the function $f(x) = 1/x$ gives functions that can be written as ratios of polynomial functions. A study of rational expressions shows that these expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression **(A-APR.D.7)**. Students apply these operations to simplify rational expressions and go on to graphing rational functions, identifying zeros and asymptotes, and analyzing end behavior **(F-IF.C.7d)**.

The module ends with Topic C in which students study inverse functions, being careful to understand when inverse functions do and do not exist, working to restrict the domain of a function to produce an invertible function. They compare and create different representations of functions including tables and graphs **(F-IF.C.9)**. Students compose functions to verify that one function is the inverse of another and work with tables of data to identify features of inverse functions **(F-BF.B.4b, F-BF.B.4c, F-BF.B.4d)**. Special emphasis is given to the inverse relationship between exponential and logarithmic functions **(F-BF.B.5).**

Rational and Exponential Functions

Unit 3

Subject: Mathematics

Grade/Course: Grade 12 / Precalculus

Pacing: 25 days

Unit of Study: Unit 3: Rational and Exponential Functions

Priority Standards:

Use complex numbers in polynomial identities and equations

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. For example, rewrite x2 + 4 as (x + 2i)(x – 2i).

N-CN.C.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Use polynomial identities to solve problems

A-APR.C.5 (+) Know and apply the Binomial Theorem for the expansion of (x + y)n in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

Rewrite rational expressions

A-APR.D.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Analyze functions using different representations

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph by hand in simple cases and using technology for more complicated cases.

d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

F-IF.C.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

Build a function that models a relationship between two quantities

F-BF.A.1 Write a function that describes a relationship between two quantities.

c. (+) Compose functions. For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.

Build new functions from existing functions

F-BF.B.4 Find inverse functions.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

F-BF.B.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Explain volume formulas and use them to solve problems

G-GMD.A.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

Translate between the geometric description and the equation for a conic section.

G-GPE.A.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Foundational Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

N-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

N-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.

Solve systems of equations.

A-REI.C.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Experiment with transformations in the plane.

G-CO.A.2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not.

G-CO.A.4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A.5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

E**xtend the domain of trigonometric functions using the unit circle.**

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for ▯/3, ▯/4 and ▯/6, and use the unit circle to express the values of sine, cosine, and tangent for ▯– ▯, ▯ + ▯, and 2▯– ▯ in terms of their values for ▯, where ▯ is any real number.

Prove and apply trigonometric identities.

F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find sin(θ), cos(θ), or tan(θ) given sin(θ), cos(θ), or tan(θ) and the quadrant of the angle.

Math Practice Standards:

MP.3 Construct viable arguments and critique the reasoning of others. Students construct arguments and critique the reasoning of others when making conjectures about the roots of polynomials (a polynomial of degree 3 will have three roots) and solving problems by applying algebraic properties. Students determine the domain and range of rational functions and reason what effect these restrictions will have on the graph of the rational function. Students use reasoning to argue that restricting the domain of a function will allow for the construction of an inverse function.

MP.7 Look for and make use of structure. Students use polynomial identities to determine roots of polynomials and square roots of complex numbers. They relate the structure of rational expressions to the graphs of rational functions by studying transformations of these graphs. Students determine the relationship between functions and their inverses.

MP.8 Look for and express regularity in repeated reasoning. Students use prior knowledge of the fundamental theorem of algebra to justify the number of roots of unity. Students develop understanding of the binomial theorem through repeated binomial expansions and connecting their observations to patterns in Pascal's Triangle. In performing and reasoning about several computations with fractional expressions, students extend the properties of rational numbers to rational expressions.

"Unwrapped" Standards

N-CN.C.8 (+) Extend polynomial identities to the complex numbers. For example, rewrite x2 + 4 as (x + 2i)(x – 2i).

N-CN.C.9 (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

A-APR.C.5 (+) Know and apply the Binomial Theorem for the expansion of (x + y)n in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

A-APR.D.7 (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph by hand in simple cases and using technology for more complicated cases.

d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior.

F-IF.C.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.

F-BF.A.1 Write a function that describes a relationship between two quantities.

c. (+) Compose functions. For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.

F-BF.B.4 Find inverse functions.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

F-BF.B.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

G-GMD.A.2 (+) Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

G-GPE.A.3 (+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.

Concepts (What Students Need to Know)	**Skills (What Students Need to Be Able to Do)**
Polynomial identities.	Extend (L4) - to complex numbers
Fundamental Theorem of Algebra.	Know (L1) Demonstrate (L1) - for quadratic polynomials
Binomial Theorem for the expansion of $(x + y)^n$.	Know (L1) Apply (L3)
Closure of system of rational expressions.	Compare (L2) - to system of rational numbers
Rational expressions.	Add, Subtract, Multiply, Divide (L1)
Functions expressed symbolically.	Graph (L2)
Key features of graphs of functions,	Identify (L1)
Properties of two functions expressed in different formats.	Compare (L2)
Functions that describe the relationship between two quantities.	Create (L4) Compose (L3) (composition of two functions)
Inverse functions.	Find (L1) Verify (L3)
Domain restrictions.	Use (L1) - to invert non-invertible functions
Inverse relationship between exponents and logarithms	Understand (L2) Solve (L2)- problems using relationship
Formula for the volume of a sphere and other solid figures.	Defend (L4) - using Cavalieri's principle

Equations of ellipses and hyperbolas.	Derive (L4)

Essential Questions	Big ideas
How are algebraic expressions used to analyze and solve problems? What characteristics of problems would determine how to model the situation and develop a problem solving strategy? How does explaining the process help to understand a problem's solution better? When and why is it necessary to follow set rules/procedures/properties when manipulating numeric or algebraic expressions?	Algebraic expressions and equations generalize relationships from specific cases. Relationships can be described and generalizations made for mathematical situations that have numbers or objects that repeat in predictable ways. Multiple representations may be used to model given real world relationships. A problem solver understands what has been done, knows why the process was appropriate, and can support it with reasons and evidence. Rules of arithmetic and algebra can be used together with notions of equivalence to transform equations and inequalities.

Assessments		
Common Formative Pre-Assessments	Progress Monitoring Checks – "Dipsticks"	Common Formative Mid and or Post-Assessments Resources
Pretest vocabulary Ellipse Horizontal Asymptote Hyperbola Vertical Asymptote	Post-test vocabulary Opening Exercise - Give again and reflect on results from first administration prior to the unit.	Type: Mid-Module Assessment Task Administered: After Topic B Format: Constructed response with rubric

Familiar Terms and Symbols Complex Numbers Domain Exponential Function Inverse Functions Logarithmic Function Polar Form Sphere Transformation Volume Conduct opening exercise Use exit ticket as pre-assessment and post where applicable	Exploratory Challenge Exit Ticket Student Conferences IXL Math	Standards Addressed: N-CN.C.8, N-CN.C.9, A-APR.C.5, G-GMD.A.2, G-GPE.A.3 Type: End-of-Module Assessment Task Administered: After Topic C Format: Constructed response with rubric Standards Addresses: A-APR.C.7, F-IF.C.7d, F-IF.C.9, F-BF.A.1c, F-BF.B.4b, F-BF.B.4c, F-BF.B.4d, F-BF.B.5

Performance Task
Performance tasks are to be created with teacher input throughout the year. A sample of a possible performance task detailed in Unit 4.
Engaging Learning Experiences
Engaging learning experiences are to be created with teacher input throughout the year. A sample of an engaging scenario is included in Unit 4.

Instructional Resources

Suggested Tools and Representations

- Graphing calculator
- ixl.com (Math)
- flippedmath.com (Precalculus)
- Geometer's Sketchpad software
- Wolfram Alpha software
- Geogebra Software

Lesson plans for all modules within the unit, that are compatible with this curriculum, can be found on EngageNY. The link below allows access to all lessons in Precalculus. (Just scroll down once you get there.) https://www.engageny.org/content/precalculus-and-advanced-topics (See Appendix A for an example.)

Instructional Strategies	Meeting the Needs of All Students
21st Century Skills Critical thinking and problem solving Collaboration and leadership Agility and Adaptability Effective oral and written communication Accessing and analyzing information **Marzano's Strategies** Identifying Similarities and Differences Reinforcing Effort and Providing Recognition Nonlinguistic Representations Homework and Practice Cooperative Learning Setting Objectives and Providing Feedback	The modules that make up Precalculus propose that the components of excellent math instruction do not change based on the audience. That said, there are specific resources included within this curriculum to highlight strategies that can provide critical access for all students. Researched-based Universal Design for Learning (UDL) has provided a structure for thinking about how to meet the needs of diverse learners. Broadly speaking, that structure asks teachers to consider multiple means of representation; multiple means of action and expression; and multiple means of engagement. Tables at the end of this section offer suggested scaffolds, utilizing this framework, for Students with Disabilities, Students Performing above Grade Level, and Students Performing below Grade Level. UDL offers ideal settings for multiple entry points for students and minimizes instructional barriers to learning. Teachers will note that many of the suggestions on a chart will be applicable to other students and overlapping populations. It is important to note that although the scaffolds/accommodations integrated into the course might change how a learner accesses information and demonstrates learning; they do not substantially alter the instructional level, content, or performance criteria. Rather, they provide students with choices in how they access content and demonstrate their knowledge and ability. **Provide Multiple Means of Representation**

	Teach from simple to complex, moving from concrete to abstract at the student's pace. Clarify, compare, and make connections to math words in discussion, particularly during and after practice. Partner key words with visuals and gestures. Connect language with concrete and pictorial experiences. Couple teacher-talk with "math-they-can-see," such as models. Let students use models and gestures to calculate and explain. For example, a student searching to define "multiplication" may model groups of 6 with drawings or concrete objects and write the number sentence to match. Teach students how to ask questions (such as "Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters," such as: "I agree because…" "Can you explain how you solved it?" "I noticed that…" "Your solution is different from/ the same as mine because…" "My mistake was to…" Couple number sentences with models. Enlarge sprint print for visually impaired learners. Use student boards to work on one calculation at a time. Invest in or make math picture dictionaries or word walls. **<u>Provide Multiple Means of Action and Expression</u>** Provide a variety of ways to respond: oral; choral; student boards; concrete models, pictorial models; pair share; small group share. For example: Use student boards to adjust "partner share" for deaf and hard-of-hearing students. Partners can jot questions and answers to one another on slates. Use vibrations or visual signs (such as clap, rather than a snap or "show") to elicit responses from deaf/hard of hearing students. Vary choral response with written response on student boards to ease linguistic barriers. Support oral or written response with sentence frames, such as "______ is ____ hundreds, ____ tens, and ____ ones. Adjust oral fluency games by using student and teacher boards or hand signals, such as showing the sum with fingers. Use visual signals or vibrations to elicit responses. Adjust wait time for interpreters of deaf and hard-of-hearing students.

	Select numbers and tasks that are “just right” for learners. Model each step of the algorithm before students begin. Give students a chance to practice the next day’s sprint beforehand. Give students a few extra minutes to process the information before giving the signal to respond. Assess by multiple means, including “show and tell” rather than written. Elaborate on the problem-solving process. Read word problems aloud. Post a visual display of the problem-solving process. Have students check off or highlight each step as they work. Talk through the problem-solving process step-by-step to demonstrate thinking process. Before students solve, ask questions for comprehension, such as, “What unit are we counting? What happened to the units in the story?” Teach students to use self-questioning techniques, such as, “Does my answer make sense?” Concentrate on goals for accomplishment within a time frame as opposed to a task frame. Extend time for task. Guide students to evaluate process and practice. Have students ask, “How did I improve? What did I do well?” Focus on students’ mathematical reasoning (i.e., their ability to make comparisons, describe patterns, generalize, explain conclusions, specify claims, and use models), not their accuracy in language. **<u>Provide Multiple Means of Engagement</u>** Make eye-to-eye contact and keep teacher-talk clear and concise. Speak clearly when checking answers for sprints and problems. Check frequently for understanding (e.g., ‘show’). Listen intently in order to uncover the math content in the students’ speech. Use non-verbal signals, such as “thumbs-up.” Assign a buddy or a group to clarify directions or process. Teach in small chunks so students get a lot of practice with one step at a time. Know, use, and make the most of Deaf culture and sign language. Use songs, rhymes, or rhythms to help students remember key concepts, such as “Add your ones up first/Make a bundle if you can!” Point to visuals and captions while speaking, using your

	hands to clearly indicate the image that corresponds to your words. Incorporate activity. Get students up and moving, coupling language with motion. Make the most of the fun exercises for activities like sprints and fluencies. Celebrate improvement. Intentionally highlight student math success frequently. Follow predictable routines to allow students to focus on content rather than behavior. Allow “everyday” and first language to express math understanding. Re-teach the same concept with a variety of fluency games. Allow students to lead group and pair-share activities. Provide learning aids, such as calculators and computers, to help students focus on conceptual understanding

New Vocabulary	Students Achieving Below Standard	Students Achieving Above Standard
New or Recently Introduced Terms **Ellipse** - the set of all points in a plane such that the sum of the distances from two points (foci) to any point on the line is constant. **Horizontal Asymptote** - Let □ be a real number. The line given by the equation □ = □ is a horizontal asymptote of the graph of □ = □(□) if at least one of the following statements is true. (1) As □ approaches infinity, □(□) approaches □. (2) As □ approaches negative infinity, □(□) approaches □. **Hyperbola** - the set of points in a plane whose distances to two fixed points □ and □, called the foci, have a constant difference. **Vertical Asymptote** - Let □ be a real number. The line given by	<u>**Provide Multiple Means of Representation**</u> Model problem-solving sets with drawings and graphic organizers (e.g., bar or tape diagram), giving many examples and visual displays. Guide students as they select and practice using their own graphic organizers and models to solve. Use direct instruction for vocabulary with visual or concrete representations. Use explicit directions with steps and procedures enumerated. Guide students through initial practice promoting gradual independence. “I do, we do, you do.” Use alternative methods of delivery of instruction such as recordings and videos that can be accessed	The following provides a bank of suggestions within the Universal Design for Learning framework for accommodating students who are above grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used. <u>**Provide Multiple Means of Representation**</u> Teach students how to ask questions (such as,“Do you agree?” and “Why do you think so?”) to extend “think-pair-share” conversations. Model and post conversation “starters,” such as: “I agree because...” “Can you explain how you solved it?” “I noticed that...”“Your solution is different from/ the same as mine because...”“My mistake was to...”

the equation ⍰ = ⍰ is a vertical asymptote of the graph of ⍰ = ⍰(⍰) if at least one of the following statements is true. (1) As ⍰ approaches ⍰, ⍰(⍰) approaches infinity. (2) As ⍰ approaches ⍰, ⍰(⍰) approaches negative infinity. **Familiar Terms and Symbols** Complex Numbers Domain Exponential Function Inverse Functions Logarithmic Function Polar Form Sphere Transformation Volume	independently or repeated if necessary. Scaffold complex concepts and provide leveled problems for multiple entry points. **<u>Provide Multiple Means of Action and Expression</u>** First use manipulatives or real objects (such as dollar bills), then make transfer from concrete to pictorial to abstract. Have students restate their learning for the day. Ask for a different representation in the restatement. 'Would you restate that answer in a different way or show me by using a diagram?' Encourage students to explain their thinking and strategy for the solution. Choose numbers and tasks that are "just right" for learners but teach the same concepts. Adjust numbers in calculations to suit learner's levels. For example, change 429 divided by 2 to 400 divided by 2 or 4 divided by 2. **<u>Provide Multiple Means of Engagement</u>** Clearly model steps, procedures, and questions to ask when solving. Cultivate peer-assisted learning interventions for instruction (e.g., dictation) and practice, particularly	Incorporate written reflection, evaluation, and synthesis. Allow creativity in expression and modeling solutions. **<u>Provide Multiple Means of Action and Expression</u>** Encourage students to explain their reasoning both orally and in writing. Extend exploration of math topics by means of challenging games, puzzles, and brain teasers. Offer choices of independent or group assignments for early finishers. Encourage students to notice and explore patterns and to identify rules and relationships in math. Have students share their observations in discussion and writing (e.g., journaling). Foster their curiosity about numbers and mathematical ideas. Facilitate research and exploration through discussion, experiments, internet searches, trips, etc. Have students compete in a secondary simultaneous competition, such as skip- counting by 75s, while peers are completing the sprint. Let students choose their mode of response: written, oral, concrete, pictorial, or abstract. Increase the pace. Offer two word problems to solve, rather than one.

	for computation work (e.g., peer modeling). Have students work together to solve and then check their solutions. Teach students to ask themselves questions as they solve: Do I know the meaning of all the words in this problem?; What is being asked?; Do I have all of the information I need?; What do I do first?; What is the order to solve this problem? What calculations do I need to make? Practice routine to ensure smooth transitions. Set goals with students regarding the type of math work students should complete in 60 seconds. Set goals with the students regarding next steps and what to focus on next. Reinforce foundational standards (listed after priority standards) for the unit.	Adjust difficulty level by increasing the number of steps (e.g., change a one-step problem to a two-step problem). Adjust difficulty level by enhancing the operation (e.g., addition to multiplication), increasing numbers to millions, or decreasing numbers to decimals/fractions. Let students write word problems to show mastery and/or extension of the content. **<u>Provide Multiple Means of Engagement</u>** Push student comprehension into higher levels of Bloom's Taxonomy with questions such as: "What would happen if...?" "Can you propose an alternative...?" "How would you evaluate...?" "What choice would you have made...?" Ask "Why?" and "What if?" questions. Celebrate improvement in completion time (e.g., Sprint A completed in 45 seconds and Sprint B completed in 30 seconds). Make the most of the fun exercises for practicing skip-counting. Accept and elicit student ideas and suggestions for ways to extend games. Cultivate student persistence in

		problem-solving and do not neglect their need for guidance and support.

Trigonometry

Overview

Trigonometry was introduced in Geometry through a study of right triangles. In Algebra II, work was conducted on extending basic trigonometry to the domain of all real numbers via the unit circle. This module revisits, unites, and expands further those ideas and introduces new tools for solving geometric and modeling problems through the power of trigonometry.

Topic A helps students recall how to use special triangles positioned within the unit circle to determine geometrically the values of sine, cosine, and tangent at special angles. The unit circle is then used to express the values of sine, cosine, and tangent for □ − □, □ + □, and 2□ − □ in terms of their values for □, where □□ is any real number **(F-TF.A.3)** and to explain the periodicity of the trigonometric functions and their symmetries **(F-TF.A.4).** Students develop the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems **(F-TF.C.9)** and to model geometric phenomena. Students also discuss the construction of tangent lines to circles **(G-C.A.4)** and revisit the geometric origins of the tangent function. Student exploration of tangents through a paper folding and compass activity is then used in modeling waves, waves traveling together, and wave patterns generated by musical instruments.

Students derive sophisticated applications of the trigonometric functions in Topic B including: the area formula for a general triangle, □ = ½ □*b* sin(□) **(G-SRT.D.9)**, the law of sines, the law of cosines, and Heron's formula. They use previous knowledge and apply their understanding of the Pythagorean theorem and oblique triangles to discover these formulas while analyzing patterns. Finally, as students investigate force diagrams and paths across rivers, they solve survey and elevation problems and revisit vectors **(G-SRT.D.10, G-SRT.D.11)**.

The graphs of the trigonometric functions are revisited in Topic C. Students visualize these graphs with the aid of the appropriate software and briefly recall how changing various parameters of a trigonometric function affects its graph. Students extend their knowledge of inverse functions to trigonometric functions **(F-TF.B.6)** as they restrict domains to create inverse trigonometric functions. These inverse functions are then used to solve trigonometric equations, evaluate their solutions using technology, and interpret these solutions in the appropriate contexts **(F-TF.B.7)**. Students determine viewing angle, line of sight, height of objects, and the angle of elevation for inclined surfaces using inverse trigonometric functions and their periodic phenomena.

Trigonometry

Unit 4

Subject: Mathematics

Grade/Course: Grade 12 / Precalculus

Pacing: 20 days

Unit of Study: Unit 4: Trigonometry

Priority Standards:

Extend the domain of trigonometric functions using the unit circle

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosine, and tangent for π–x, π+x, and 2π–x in terms of their values for x, where x is any real number.

F-TF.A.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Model periodic phenomena with trigonometric functions

F-TF.B.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

F-TF.B.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

Prove and apply trigonometric identities

F-TF.C.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Understand and apply theorems about circles

G-C.A.C.4 (+) Construct a tangent line from a point outside a given circle to the circle.

Apply trigonometry to general triangles

G-SRT.D.9 (+) Derive the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G-SRT.D.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G-SRT.D.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Foundational Standards

Extend the domain of trigonometric functions using the unit circle

F-TF.A.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

F-TF.A.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

Model periodic phenomena with trigonometric functions

F-TF.B.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

Prove and apply trigonometric identities

F-TF.C.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Understand the concept of a function and use function notation

F-IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F-IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Analyze functions using different representations

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.

e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Build new functions from existing functions

F-BF.B.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$ or $f(x) = (x + 1)/(x - 1)$ for $x \neq 1$.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read values of an inverse function from a graph or table, given that the function has an inverse.

d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Define trigonometric ratios and solve problems involving right triangles

G-SRT.C.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

G-SRT.C.7 Explain and use the relationship between the sine and cosine of complementary angles.

G-SRT.C.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

Understand and apply theorems about circles

G-C.A.2 Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

Prove geometric theorems.

G-CO.C.10 Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

Math Practice Standards:

MP.3 Construct viable arguments and critique the reasoning of others. Students construct mathematical arguments as they explain their calculations of the area of triangles leading to a new formula for area, Area = ½ *a*□ sin(□). Students explain the properties of trigonometric functions and explain construction of tangent lines. Students use the periodic nature of trigonometric functions to reason about their graphs and problem solve using inverse functions.

MP.4 Model with mathematics. Students apply sum and difference formulas in the context of modeling sound waves using trigonometric functions. Students model using trigonometric functions applying tangent lines and Law of Sines and Cosines to solve surveying problems and revisit vectors. Students investigate viewing distance, line of sight, and viewing angle using inverse trigonometric functions as well as the angle of elevation for inclined surfaces.

MP.5 Use appropriate tools strategically. Students see the unit circle as a tool to determine the values of trigonometric functions in terms of □□ and explain their periodicity and symmetry. Students use computer software and graphing calculators to graph trigonometric functions and their inverses. Students see trigonometric inverses, Law of Sines, Law of Cosines, and area formulas as tools in problem solving.

“Unwrapped” Standards

F-TF.A.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for π/3, π/4 and π/6, and use the unit circle to express the values of sine, cosine, and tangent for π–x, π+x, and 2π–x in terms of their values for x, where x is any real number.

F-TF.A.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F-TF.B.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

F-TF.B.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

F-TF.C.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

G-C.A.C.4 (+) Construct a tangent line from a point outside a given circle to the circle.

G-SRT.D.9 (+) Derive the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.

G-SRT.D.10 (+) Prove the Laws of Sines and Cosines and use them to solve problems.

G-SRT.D.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Concepts (What Students Need to Know)	**Skills (What Students Need to Be Able to Do)**
The values of sine, cosine, tangent for π/3, π/4 and π/6,	Determine (L1) - geometrically using special triangles Determine (L1) - using the unit circle
The values of sine, cosine, and tangent for π–x, π+x, and 2π–x in terms of their values for x.	
Symmetry and periodicity of trigonometric functions.	Explain (L2) - using the unit circle
Restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.	Understand (L2)
Trigonometric equations that arise in modeling contexts	Solve (L2) - using inverse functions Evaluate (L4) - using technology Interpret (L2) - in terms of context

Addition and subtraction formulas for sine, cosine, and tangent.	Prove (L4) Solve (L2) - problems with formulas
A tangent line from a point outside a given circle to the circle.	Construct (L3)
The formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle.	Derive (L3)
The Laws of Sines and Cosines.	Prove (L4) Solve (L2) Understand (L2) Apply (L3)

Essential Questions	**Big ideas**
How do geometric models describe spatial relationships?	Geometry and spatial sense offer ways to interpret and reflect on our physical environment.
When and how is mathematics used in solving real world problems?	Mathematics can be used to solve real world problems and can be used to communicate solutions. Relationships between quantities can be represented symbolically, numerically, graphically and verbally in the exploration of real world situations.
How can the knowledge of trigonometric functions be used to solve problems?	Trigonometric functions have properties that make them a better choice to model real world applications.

Assessments		
Common Formative Pre-Assessments	Progress Monitoring Checks – “Dipsticks”	Common Formative Mid and or Post-Assessments Resources
Pretest vocabulary No new terms in this unit Familiar Terms and Symbols	Post-test vocabulary Opening Exercise - Give again and reflect on results from first	Type: Mid-Module Assessment Task Administered: After Topic A Format: Constructed response

Sine Cosine Tangent Function Radian Measure Tangent Line Unit Circle Period Vector Conduct opening exercise Use exit ticket as pre-assessment and post where applicable	administration prior to the unit. Exploratory Challenge Exit Ticket Student Conferences IXL Math	with rubric Standards Addressed: F-TF.A.3, F-TF.A.4, F-TF.C.9, G-C.A.4 Type: End-of-Module Assessment Task Administered: After Topic C Format: Constructed response with rubric Standards Addresses: F-TF.B.6, F-TF.B.7, G-SRT.D.9, G-SRT.D.10, G-SRT.D.11

Performance Task

Performance tasks are to be created with teacher input throughout the year. A sample of a possible performance task detailed below.

"Sohcahtoa Mountain Blizzard" (The "engaging" scenario that accompanies this performance task is detailed following the tasks in the "Engaging Learning Experiences" section.)

Task 1
Using your trigonometric skills, calculate the steepness of each run by finding its angle of elevation. Use the space below to show your calculations and then show a summary of the results in the table.

Summary of Results:

Run	Steepness	Rating
Slow Sam		
Bobsled Bop		
Dare Devil Drop		
Avalanche Alley		
Thunder Run		

Task 2

Trisha Timid has only started learning how to snowboard. She wants to go right to the top of the mountain and then find an easy way down. She thinks she can navigate Avalanche Alley, but doesn't want to risk going on Thunder Run. She is wondering if she could cut across to the top of Slow Sam (this would be from point E to point B on the map).

a) She wants to know how long this would be. Find the length of Trisha's run from the top of the mountain if she goes down Avalanche Alley cuts across and then goes down Slow Sam.

b) What rating would the run from E to B be? Explain how you know.

Task 3

The owners of the resort, Isos and Celes Triangle want to expand and build a row of condos at the western base of the mountain. Because of the amount of snow the area gets most winters, it is important to have the pitch (steepness) of the roof of each condo at least 3:2. To make the condos appealing to skiers and boarders, the Triangles want to model the condos after their chalets, but on a larger scale. The chalets have an A-line roof that forms an isosceles triangle as shown, with the base angles at 53^{o}. The base length is 4m. The condos to be built will have a length of 7.4m.

a) What is the slant height of the roof on the chalet (to the nearest tenth of a meter)?

b) What will be the slant height of the roof on one of the condos?

Task 4

Sohcahtoa Mountain has an altitude of 800m. It is one of several that form the Scalenes Mountain Range. The mountain to the east of Sohcahtoa Mountain has a higher altitude. George Geometer was taking a break from skiing and was basking in the sun at the top of Sohcahtoa Mountain. He wondered how much higher the eastern mountain was. He was able to see down to the base of the mountain at an angle of and look up to the peak at an angle of 36^{o}.

Find out how much higher the eastern mountain is (to the nearest tenth of a meter).

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The elements of performance required by this task are:

Use special triangles to determine geometrically the values of sine, cosine, tangent.

Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles

Possible Responses Point distribution

1. Total possible points - 10
   - Calculates the steepness of each of the ski runs. - 10 points (2 each)
   - *Partial credit*
     - *Computational errors - deduct 1 (each)*
2. Total possible points - 5
   - a) Correctly calculates the length of Trisha's Run - 3 pt
   - b) Explains clearly the reasoning behind the run rating (including math references). -2
   - *Partial credit*

Computational errors in (a) - deduct 1 (each)

3. Total possible points - 4
   a) Correctly determines the slant height of the roof on the chalet. - 2
   b) Correctly determines the slant height of the roof on one of the condos. - 2
   Partial credit
   Computational error(s) in part (a) or (b) - deduct 1 (each)

4. Total possible points - 2
   Correctly determines how much higher the eastern mountain is. - 2
   Partial credit
   Computational error - deduct - 1

TOTAL POSSiBLE POiNTS = 21

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Performance Level Descriptions and Cut Scores
Performance is reported at four levels: 1 through 4, with 4 as the highest.

Level 1: Demonstrates Minimal Success (0–4 points)
The student's response shows few of the elements of performance that the task demands as defined by the CCSS. The student's work shows a minimal attempt and lack of coherence. The student fails to use appropriate tools strategically. The student is unable to make sense of the problem and apply mathematical concepts in this modeling situation.

Level 2: Performance Below Standard (5–10 points)
The student's response shows some of the elements of performance that the task demands as defined by the CCSS. The student might ignore or fail to address some of the constraints of the problem. The student may occasionally make sense of quantities or relationships in the problem. The student attempts to use some appropriate tools with limited success. The student may have trouble generalizing or applying mathematical methods in this modeling situation.

Level 3: Performance at Standard (11–15 points)
For most of the task, the student's response shows the main elements of performance that the tasks demand as defined by the CCSS with few minor errors or omissions. The student explains the problem and identifies constraints. The student makes sense of quantities and their relationships in the modeling situation. The student uses appropriate tools. The student might discern patterns or structures and make connections between representations. The student is able to make sense of the problem and apply geometric concepts to this modeling situation.

Level 4: Achieves Standards at a High Level (16–18 points)
The student's response meets the demands of nearly all of the tasks as defined by the CCSS and is organized in a coherent way. The communication is clear and precise. The body of work looks at the overall situation of the problem and process, while attending to the details. The student routinely interprets the mathematical results, applies concepts in the context of the situation, reflects on whether the results make sense and uses all appropriate tools strategically.

Engaging Learning Experiences
Engaging learning experiences are to be created with teacher input throughout the year. A sample of an engaging scenario is included below. The scenario can be used for the performance task listed above. The Sohcahtoa Mountain Ski Resort is a popular ski club, especially with snow boarders. Last weekend though, the resort had to shut down for three days because of a major blizzard. When the ski patrol went to check the runs after the blizzard, they couldn't find the rating signs that tell how difficult the runs are. They returned back to the base lodge to check the map on the computer. Unfortunately, the blizzard caused a power outage and the computer that had the files listing the steepness of each run were lost. All the ski patrol could find was a scale drawing of the mountain that outlines the runs and chairlifts and shows where the chalets and base lodge are. The length of the runs are also on the map. Howie Trig, the head of the ski patrol was not worried. He said they had all the information they needed to find the steepness of each run using their trigonometry skills.

Instructional Resources
Suggested Tools and Representations Compass Straightedge Graphing calculator ixl.com (Math) flippedmath.com (Precalculus) Wolfram Alpha Software Geometer's Sketchpad Software GeoGebra software Lesson plans for all modules within the unit, that are compatible with this curriculum, can be found on EngageNY. The link below allows access to all lessons in Precalculus. (Just scroll down once you get there.) https://www.engageny.org/content/precalculus-and-advanced-topics (See Appendix A for an example.)

Instructional Strategies	Meeting the Needs of All Students

21st Century Skills Critical thinking and problem solving Collaboration and leadership Agility and Adaptability Effective oral and written communication Accessing and analyzing information **Marzano's Strategies** Identifying Similarities and Differences Reinforcing Effort and Providing Recognition Nonlinguistic Representations Homework and Practice Cooperative Learning Setting Objectives and Providing Feedback	The modules that make up Precalculus propose that the components of excellent math instruction do not change based on the audience. That said, there are specific resources included within this curriculum to highlight strategies that can provide critical access for all students. Researched-based Universal Design for Learning (UDL) has provided a structure for thinking about how to meet the needs of diverse learners. Broadly speaking, that structure asks teachers to consider multiple means of representation; multiple means of action and expression; and multiple means of engagement. Tables at the end of this section offer suggested scaffolds, utilizing this framework, for Students with Disabilities, Students Performing above Grade Level, and Students Performing below Grade Level. UDL offers ideal settings for multiple entry points for students and minimizes instructional barriers to learning. Teachers will note that many of the suggestions on a chart will be applicable to other students and overlapping populations. It is important to note that although the scaffolds/accommodations integrated into the course might change how a learner accesses information and demonstrates learning; they do not substantially alter the instructional level, content, or performance criteria. Rather, they provide students with choices in how they access content and demonstrate their knowledge and ability. **<u>Provide Multiple Means of Representation</u>** Teach from simple to complex, moving from concrete to abstract at the student's pace. Clarify, compare, and make connections to math words in discussion, particularly during and after practice. Partner key words with visuals and gestures. Connect language with concrete and pictorial experiences. Couple teacher-talk with "math-they-can-see," such as models. Let students use models and gestures to calculate and explain. For example, a student searching to define "multiplication" may model groups of 6 with drawings or concrete objects and write the number sentence to match. Teach students how to ask questions (such as "Do you agree?" and "Why do you think so?") to extend "think-

pair-share” conversations. Model and post conversation “starters,” such as: “I agree because…” “Can you explain how you solved it?” “I noticed that…” “Your solution is different from/ the same as mine because…” “My mistake was to…”

Couple number sentences with models.

Enlarge sprint print for visually impaired learners.

Use student boards to work on one calculation at a time.

Invest in or make math picture dictionaries or word walls.

<u>Provide Multiple Means of Action and Expression</u>

Provide a variety of ways to respond: oral; choral; student boards; concrete models, pictorial models; pair share; small group share. For example: Use student boards to adjust “partner share” for deaf and hard-of-hearing students. Partners can jot questions and answers to one another on slates. Use vibrations or visual signs (such as clap, rather than a snap or “show”) to elicit responses from deaf/hard of hearing students.

Vary choral response with written response on student boards to ease linguistic barriers. Support oral or written response with sentence frames, such as “______ is ____ hundreds, ____ tens, and ____ ones.

Adjust oral fluency games by using student and teacher boards or hand signals, such as showing the sum with fingers. Use visual signals or vibrations to elicit responses.

Adjust wait time for interpreters of deaf and hard-of-hearing students.

Select numbers and tasks that are “just right” for learners.

Model each step of the algorithm before students begin.

Give students a chance to practice the next day’s sprint beforehand.

Give students a few extra minutes to process the information before giving the signal to respond.

Assess by multiple means, including “show and tell” rather than written.

Elaborate on the problem-solving process. Read word problems aloud. Post a visual display of the problem-solving process. Have students check off or highlight each step as they work. Talk through the problem-solving process step-by-step to demonstrate thinking

	process. Before students solve, ask questions for comprehension, such as, “What unit are we counting? What happened to the units in the story?” Teach students to use self-questioning techniques, such as, “Does my answer make sense?” Concentrate on goals for accomplishment within a time frame as opposed to a task frame. Extend time for task. Guide students to evaluate process and practice. Have students ask, “How did I improve? What did I do well?” Focus on students’ mathematical reasoning (i.e., their ability to make comparisons, describe patterns, generalize, explain conclusions, specify claims, and use models), not their accuracy in language. **<u>Provide Multiple Means of Engagement</u>** Make eye-to-eye contact and keep teacher-talk clear and concise. Speak clearly when checking answers for sprints and problems. Check frequently for understanding (e.g., ‘show’). Listen intently in order to uncover the math content in the students’ speech. Use non-verbal signals, such as “thumbs-up.” Assign a buddy or a group to clarify directions or process. Teach in small chunks so students get a lot of practice with one step at a time. Know, use, and make the most of Deaf culture and sign language. Use songs, rhymes, or rhythms to help students remember key concepts, such as “Add your ones up first/Make a bundle if you can!” Point to visuals and captions while speaking, using your hands to clearly indicate the image that corresponds to your words. Incorporate activity. Get students up and moving, coupling language with motion. Make the most of the fun exercises for activities like sprints and fluencies. Celebrate improvement. Intentionally highlight student math success frequently. Follow predictable routines to allow students to focus on content rather than behavior. Allow “everyday” and first language to express math understanding. Re-teach the same concept with a variety of fluency games. Allow students to lead group and pair-share activities.

	Provide learning aids, such as calculators and computers, to help students focus on conceptual understanding

New Vocabulary	Students Achieving Below Standard	Students Achieving Above Standard
New or Recently Introduced Terms No new terms in this unit Familiar Terms and Symbols Sine Cosine Tangent Function Radian Measure Tangent Line Unit Circle Period Vector	**Provide Multiple Means of Representation** Model problem-solving sets with drawings and graphic organizers (e.g., bar or tape diagram), giving many examples and visual displays. Guide students as they select and practice using their own graphic organizers and models to solve. Use direct instruction for vocabulary with visual or concrete representations. Use explicit directions with steps and procedures enumerated. Guide students through initial practice promoting gradual independence. "I do, we do, you do." Use alternative methods of delivery of instruction such as recordings and videos that can be accessed independently or repeated if necessary. Scaffold complex concepts and provide leveled problems for multiple entry points.	The following provides a bank of suggestions within the Universal Design for Learning framework for accommodating students who are above grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used. **Provide Multiple Means of Representation** Teach students how to ask questions (such as,"Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters," such as: "I agree because..." "Can you explain how you solved it?" "I noticed that...""Your solution is different from/ the same as mine because...""My mistake was to..." Incorporate written reflection, evaluation, and synthesis. Allow creativity in expression and modeling solutions. **Provide Multiple Means of Action and Expression**

	Provide Multiple Means of Action and Expression First use manipulatives or real objects (such as dollar bills), then make transfer from concrete to pictorial to abstract. Have students restate their learning for the day. Ask for a different representation in the restatement. 'Would you restate that answer in a different way or show me by using a diagram?' Encourage students to explain their thinking and strategy for the solution. Choose numbers and tasks that are "just right" for learners but teach the same concepts. Adjust numbers in calculations to suit learner's levels. For example, change 429 divided by 2 to 400 divided by 2 or 4 divided by 2. **Provide Multiple Means of Engagement** Clearly model steps, procedures, and questions to ask when solving. Cultivate peer-assisted learning interventions for instruction (e.g., dictation) and practice, particularly for computation work (e.g., peer modeling). Have students work together to solve and then check their solutions. Teach students to ask themselves questions as they solve: Do I know	Encourage students to explain their reasoning both orally and in writing. Extend exploration of math topics by means of challenging games, puzzles, and brain teasers. Offer choices of independent or group assignments for early finishers. Encourage students to notice and explore patterns and to identify rules and relationships in math. Have students share their observations in discussion and writing (e.g., journaling). Foster their curiosity about numbers and mathematical ideas. Facilitate research and exploration through discussion, experiments, internet searches, trips, etc. Have students compete in a secondary simultaneous competition, such as skip-counting by 75s, while peers are completing the sprint. Let students choose their mode of response: written, oral, concrete, pictorial, or abstract. Increase the pace. Offer two word problems to solve, rather than one. Adjust difficulty level by increasing the number of steps (e.g., change a one-step problem to a two-step problem). Adjust difficulty level by enhancing the operation (e.

	the meaning of all the words in this problem?; What is being asked?; Do I have all of the information I need?; What do I do first?; What is the order to solve this problem? What calculations do I need to make? Practice routine to ensure smooth transitions. Set goals with students regarding the type of math work students should complete in 60 seconds. Set goals with the students regarding next steps and what to focus on next. Reinforce foundational standards (listed after priority standards) for the unit.	g., addition to multiplication), increasing numbers to millions, or decreasing numbers to decimals/fractions. Let students write word problems to show mastery and/or extension of the content. **<u>Provide Multiple Means of Engagement</u>** Push student comprehension into higher levels of Bloom's Taxonomy with questions such as: "What would happen if...?""Can you propose an alternative...?""How would you evaluate...?""What choice would you have made...?" Ask "Why?" and "What if?" questions. Celebrate improvement in completion time (e.g., Sprint A completed in 45 seconds and Sprint B completed in 30 seconds). Make the most of the fun exercises for practicing skip-counting. Accept and elicit student ideas and suggestions for ways to extend games. Cultivate student persistence in problem-solving and do not neglect their need for guidance and support.

Probability and Statistics

Overview

In this module, students build on their understanding of probability developed in previous grades. In Topic A, the multiplication rule for independent events introduced in Grade 11 is generalized to a rule that can be used to calculate the probability of the intersection of two events in situations where the two events are not independent. In this topic, students are also introduced to three techniques for counting outcomes—the fundamental counting principle, permutations, and combinations. These techniques are then used to calculate probabilities, and these probabilities are interpreted in context **(S-CP.B.8, S-CP.B.9).**

In Topic B, students study probability distributions for discrete random variables **(S-MD.A.1).** They develop an understanding of the information that a probability distribution provides and interpret probabilities from the probability distribution of a discrete random variable in context **(S-MD.A.2).** For situations where the probabilities associated with a discrete random variable can be calculated given a description of the random variable, students determine the probability distribution **(S-MD.A.3).** Students also see how empirical data can be used to approximate the probability distribution of a discrete random variable **(S-MD.A.4).** This topic also introduces the idea of expected value, and students calculate and interpret the expected value of discrete random variables in context.

Topic C is a capstone topic for this module, where students use what they have learned about probability and expected value to analyze strategies and make decisions in a variety of contexts **(S-MD.B.5, S-MD.B.6, S-MD.B.7).** Students use probabilities to make a fair decision and explain how to make fair and “unfair” decisions. Students analyze simple games of chance as they calculate and interpret the expected payoff in context. They make decisions based on expected values in problems with business, medical, and other contexts. They also examine and interpret what it means for a game to be fair. Interpretation and explanations of expected values are important outcomes for Topic C.

Probability and Statistics

Unit 5

Subject: Mathematics

Grade/Course: Grade 12 / Precalculus

Pacing: 25 days

Unit of Study: Unit 5: Probability and Statistics

Priority Standards:

Use the rules of probability to compute probabilities of compound events in a uniform probability model

S-CP.B.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

S-CP.B.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Calculate expected values and use them to solve problems

S-MD.A.1 (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

S-MD.A.2 (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

S-MD.A.3 (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.

S-MD.A.4 (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?

Use probability to evaluate outcomes of decisions

S-MD.B.5 (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

a. Find the expected payoff for a game of chance. For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.

b. Evaluate and compare strategies on the basis of expected values. For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.

S-MD.B.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

S-MD.B.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Foundational Standards

Understand independence and conditional probability and use them to interpret data

S-CP.A.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

S-CP.A.2 Understand that two events □ and □ are independent if the probability of □ and □ occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

S-CP.A.3 Understand the conditional probability of □ given □ as □(□ and □)/□(□), and interpret independence of □ and □ as saying that the conditional probability of □ given □ is the same as the probability of □, and the conditional probability of □ given □ is the same as the probability of □.

S-CP.A.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.

S-CP.A.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.

Use the rules of probability to compute probabilities of compound events in a uniform probability model

S-CP.B.6 Find the conditional probability of □ given □ as the fraction of □'s outcomes that also belong to □, and interpret the answer in terms of the model.

S-CP.B.7 Apply the Addition Rule, □(□ or □)– □(□) + □(□)– □(□ and □), and interpret the answer in terms of the model.

Math Practice Standards:

MP.2 Reason abstractly and quantitatively. Students interpret probabilities calculated using the addition rule and the general multiplication rule. They use permutations and combinations to calculate probabilities and interpret them in context. Students also explain the meaning of the expected value of a random variable as a long-run average and connect this interpretation to the given context.

MP.3 Construct viable arguments and critique the reasoning of others. Students construct arguments in distinguishing between situations involving combinations and those involving permutations. Students use permutations and combinations to calculate probabilities and evaluate decisions based on probabilities. Students also use expected values to analyze games of chance and to evaluate whether a game is "fair." Students design, compare, and evaluate games of chance that they construct, comparing their games to the games of other students based on probabilities and expected values. They analyze strategies based on probability. For example, students use expected value to explain which of two plans yields the largest earnings for an insurance company.

MP.4 Model with mathematics. Students develop a probability distribution for a random variable by finding the theoretical probabilities. Students model probability distributions by estimating probabilities empirically. They use probabilities to make and justify decisions. Throughout the module, students use statistical ideas to explain and solve real-world problems. For example, given the probability of finding a female egg in a nest, students determine a discrete probability distribution for the number of male eggs in the nest.

MP.5 Use appropriate tools strategically. Students use technology to carry out simulations in order to estimate probabilities empirically. For example, students use technology to simulate a dice-tossing game and generate random numbers to simulate the flavors in a pack of cough drops. They use technology to graph a probability distribution and to calculate expected values. Students come to view discrete probability distributions as tools that can be used to understand real-world situations and solve problems.

MP.8 Look for and express regularity in repeated reasoning. Students use simulations to observe the long-run behavior of a random variable, using the results of the simulations to estimate probabilities.

"Unwrapped" Standards

S-CP.B.8 (+) Apply the general Multiplication Rule in a uniform probability model, P(A and B) = P(A)P(B|A) = P(B)P(A|B), and interpret the answer in terms of the model.

S-CP.B.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

S-MD.A.1 (+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.

S-MD.A.2 (+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.

S-MD.A.3 (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.

S-MD.A.4 (+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?

S-MD.B.5 (+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values.

a. Find the expected payoff for a game of chance. For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.

b. Evaluate and compare strategies on the basis of expected values. For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.

S-MD.B.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

S-MD.B.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Concepts (What Students Need to Know)	**Skills (What Students Need to Be Able to Do)**
Multiplication Rule in a uniform probability model.	Apply (L2) - Interpret (L2) - the solution
Probabilities of compound events.	Compute (L1) - using combinations and permutations Solve (L2) - problems
Probability distribution.	Define (L2) - by assigning a numerical value to each event in a sample space Graph (L2)
The expected value of a random variable.	

	Calculate (L1) Interpret (L2) - as the mean of a probability distribution
Probability distribution for a random variable.	Develop (L3) - using theoretical probabilities Develop (L3) - using probabilities assigned empirically
Expected values.	Find (L1)
Expected payoff for a game of chance.	Find (L1)
Strategies on the basis of expected values.	Compare (L2) Evaluate (L4)
Fair decisions.	Make (L4) - using probabilities
Decisions and strategies.	Analyze (L4) - using probability concepts

Essential Questions	**Big ideas**
Why is data collected and analyzed?	Specific data collected from a sample is used to predict and understand properties of an entire population.
How is data used to influence opinions and decision making?	The way that data is collected , organized and displayed influences interpretation.
How can predictions be made using collected data?	The probability of an event’s occurrence can be predicted with varying degrees of confidence.

Assessments		
Common Formative Pre-Assessments	Progress Monitoring Checks – “Dipsticks”	Common Formative Mid and or Post-Assessments Resources
Pretest vocabulary **New terms** Combination of □ items selected from a set of □ distinct items Continuous random variables Discrete random variables Empirical probability Expected value of a random variable Fundamental counting principle General multiplication rule Long-run behavior of a random variable Permutation of □ items selected from a set of □ distinct items Probability distribution Probability distribution of a discrete random variable Random variable Theoretical probability Uniform probability model **Familiar Terms and Symbols** Chance experiment Complement of an event Event Intersection of events Sample space Union of events Conduct opening exercise Use exit ticket as pre-assessment and post where applicable	Post-test vocabulary Opening Exercise - Give again and reflect on results from first adminstration prior to the unit. Exploratory Challenge Exit Ticket Student Confetences IXL Math	Type: Mid-Module Assessment Task Administered: After Topic B Format: Constructed response with rubric Standards Addressed: S-CP.B.8, S-CP.B.9, S-MD.A.1, S-MD.A.2, S-MD.A.3, S-MD.A.4 Type: End-of-Module Assessment Task Administered: After Topic C Format: Constructed response with rubric Standards Addresses: S-CP.B.8, S-CP.B.9, S-MD.A.2, S-MD.A.3, S-MD.B.5, S-MD.B.6, S-MD.B.7

Performance Task
Performance tasks are to be created with teacher input throughout the year. A sample of a possible performance task detailed in Unit 4.

Engaging Learning Experiences
Engaging learning experiences are to be created with teacher input throughout the year. A sample of an engaging scenario is included in Unit 4.

Instructional Resources
Suggested Tools and Representations Graphing calculator ixl.com (Math) flippedmath.com (Precalculus) Random number software Random number tables Two-way frequency tables Lesson plans for all modules within the unit, that are compatible with this curriculum, can be found on EngageNY. The link below allows access to all lessons in Precalculus. (Just scroll down once you get there.) https://www.engageny.org/content/precalculus-and-advanced-topics (See Appendix A for an example.)

Instructional Strategies	Meeting the Needs of All Students

21st Century Skills Critical thinking and problem solving Collaboration and leadership Agility and Adaptability Effective oral and written communication Accessing and analyzing information **Marzano's Strategies** Identifying Similarities and Differences Reinforcing Effort and Providing Recognition Nonlinguistic Representations Homework and Practice Cooperative Learning Setting Objectives and Providing Feedback	The modules that make up Precalculus propose that the components of excellent math instruction do not change based on the audience. That said, there are specific resources included within this curriculum to highlight strategies that can provide critical access for all students. Researched-based Universal Design for Learning (UDL) has provided a structure for thinking about how to meet the needs of diverse learners. Broadly speaking, that structure asks teachers to consider multiple means of representation; multiple means of action and expression; and multiple means of engagement. Tables at the end of this section offer suggested scaffolds, utilizing this framework, for Students with Disabilities, Students Performing above Grade Level, and Students Performing below Grade Level. UDL offers ideal settings for multiple entry points for students and minimizes instructional barriers to learning. Teachers will note that many of the suggestions on a chart will be applicable to other students and overlapping populations. It is important to note that although the scaffolds/accommodations integrated into the course might change how a learner accesses information and demonstrates learning; they do not substantially alter the instructional level, content, or performance criteria. Rather, they provide students with choices in how they access content and demonstrate their knowledge and ability. **Provide Multiple Means of Representation** Teach from simple to complex, moving from concrete to abstract at the student's pace. Clarify, compare, and make connections to math words in discussion, particularly during and after practice. Partner key words with visuals and gestures. Connect language with concrete and pictorial experiences. Couple teacher-talk with "math-they-can-see," such as models. Let students use models and gestures to calculate and explain. For example, a student searching to define "multiplication" may model groups of 6 with drawings or concrete objects and write the number sentence to match. Teach students how to ask questions (such as "Do you agree?" and "Why do you think so?") to extend "think-

	pair-share" conversations. Model and post conversation "starters," such as: "I agree because…" "Can you explain how you solved it?" "I noticed that…" "Your solution is different from/ the same as mine because…" "My mistake was to…" Couple number sentences with models. Enlarge sprint print for visually impaired learners. Use student boards to work on one calculation at a time. Invest in or make math picture dictionaries or word walls. **<u>Provide Multiple Means of Action and Expression</u>** Provide a variety of ways to respond: oral; choral; student boards; concrete models, pictorial models; pair share; small group share. For example: Use student boards to adjust "partner share" for deaf and hard-of-hearing students. Partners can jot questions and answers to one another on slates. Use vibrations or visual signs (such as clap, rather than a snap or "show") to elicit responses from deaf/hard of hearing students. Vary choral response with written response on student boards to ease linguistic barriers. Support oral or written response with sentence frames, such as "______ is ____ hundreds, ____ tens, and ____ ones. Adjust oral fluency games by using student and teacher boards or hand signals, such as showing the sum with fingers. Use visual signals or vibrations to elicit responses. Adjust wait time for interpreters of deaf and hard-of-hearing students. Select numbers and tasks that are "just right" for learners. Model each step of the algorithm before students begin. Give students a chance to practice the next day's sprint beforehand. Give students a few extra minutes to process the information before giving the signal to respond. Assess by multiple means, including "show and tell" rather than written. Elaborate on the problem-solving process. Read word problems aloud. Post a visual display of the problem-solving process. Have students check off or highlight each step as they work. Talk through the problem-solving process step-by-step to demonstrate thinking

	process. Before students solve, ask questions for comprehension, such as, “What unit are we counting? What happened to the units in the story?” Teach students to use self-questioning techniques, such as, “Does my answer make sense?” Concentrate on goals for accomplishment within a time frame as opposed to a task frame. Extend time for task. Guide students to evaluate process and practice. Have students ask, “How did I improve? What did I do well?” Focus on students’ mathematical reasoning (i.e., their ability to make comparisons, describe patterns, generalize, explain conclusions, specify claims, and use models), not their accuracy in language. **<u>Provide Multiple Means of Engagement</u>** Make eye-to-eye contact and keep teacher-talk clear and concise. Speak clearly when checking answers for sprints and problems. Check frequently for understanding (e.g., ‘show’). Listen intently in order to uncover the math content in the students’ speech. Use non-verbal signals, such as “thumbs-up.” Assign a buddy or a group to clarify directions or process. Teach in small chunks so students get a lot of practice with one step at a time. Know, use, and make the most of Deaf culture and sign language. Use songs, rhymes, or rhythms to help students remember key concepts, such as “Add your ones up first/Make a bundle if you can!” Point to visuals and captions while speaking, using your hands to clearly indicate the image that corresponds to your words. Incorporate activity. Get students up and moving, coupling language with motion. Make the most of the fun exercises for activities like sprints and fluencies. Celebrate improvement. Intentionally highlight student math success frequently. Follow predictable routines to allow students to focus on content rather than behavior. Allow “everyday” and first language to express math understanding. Re-teach the same concept with a variety of fluency games. Allow students to lead group and pair-share activities.

	Provide learning aids, such as calculators and computers, to help students focus on conceptual understanding

New Vocabulary	Students Achieving Below Standard	Students Achieving Above Standard
New or Recently Introduced Terms **Combination of ⍰ items selected from a set of ⍰ distinct items** - an unordered set of ⍰ items selected from a set of ⍰ distinct items **Continuous random variables** - a random variable for which the possible values form an entire interval along the number line **Discrete random variables** - a random value for which the possible values are isolated points along the number line **Empirical probability** - a probability that has been estimated by observing a large number of outcomes of a chance experiment or values of a random variable **Expected value of a random variable** - the long-run average value expected over a large number of observations of the value of a random variable **Fundamental counting principle** **General multiplication rule** - a probability rule for calculating the probability of the intersection of two events **Long-run behavior of a random variable** - the behavior of the random variable over a very long	**<u>Provide Multiple Means of Representation</u>** Model problem-solving sets with drawings and graphic organizers (e.g., bar or tape diagram), giving many examples and visual displays. Guide students as they select and practice using their own graphic organizers and models to solve. Use direct instruction for vocabulary with visual or concrete representations. Use explicit directions with steps and procedures enumerated. Guide students through initial practice promoting gradual independence. "I do, we do, you do." Use alternative methods of delivery of instruction such as recordings and videos that can be accessed independently or repeated if necessary. Scaffold complex concepts and provide leveled problems for multiple entry points.	The following provides a bank of suggestions within the Universal Design for Learning framework for accommodating students who are above grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used. **<u>Provide Multiple Means of Representation</u>** Teach students how to ask questions (such as,"Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters," such as: "I agree because..." "Can you explain how you solved it?" "I noticed that...""Your solution is different from/ the same as mine because...""My mistake was to..." Incorporate written reflection, evaluation, and synthesis. Allow creativity in expression and modeling solutions. **<u>Provide Multiple Means of Action and Expression</u>**

sequence of observations **Permutation of ⍰ items selected from a set of ⍰ distinct items** - an ordered sequence of ⍰ items selected from a set of ⍰ distinct items **Probability distribution** - a table or graph that provides information about the long-run behavior of a random variable **Probability distribution of a discrete random variable** - a table or graph that specifies the possible values of the random variable and the associated probabilities **Random variable** - a variable whose possible values are based on the outcome of a random event **Theoretical probability** - a probability calculated by assigning a probability to all possible outcomes in the sample space for a chance experiment **Uniform probability model** - a probability distribution that assigns equal probability to each possible outcome of a chance experiment Familiar Terms and Symbols Chance experiment Complement of an event Event Intersection of events Sample space Union of events	**<u>Provide Multiple Means of Action and Expression</u>** First use manipulatives or real objects (such as dollar bills), then make transfer from concrete to pictorial to abstract. Have students restate their learning for the day. Ask for a different representation in the restatement. 'Would you restate that answer in a different way or show me by using a diagram?' Encourage students to explain their thinking and strategy for the solution. Choose numbers and tasks that are "just right" for learners but teach the same concepts. Adjust numbers in calculations to suit learner's levels. For example, change 429 divided by 2 to 400 divided by 2 or 4 divided by 2. **<u>Provide Multiple Means of Engagement</u>** Clearly model steps, procedures, and questions to ask when solving. Cultivate peer-assisted learning interventions for instruction (e.g., dictation) and practice, particularly for computation work (e.g., peer modeling). Have students work together to solve and then check their solutions. Teach students to ask themselves questions as they solve: Do I know	Encourage students to explain their reasoning both orally and in writing. Extend exploration of math topics by means of challenging games, puzzles, and brain teasers. Offer choices of independent or group assignments for early finishers. Encourage students to notice and explore patterns and to identify rules and relationships in math. Have students share their observations in discussion and writing (e.g., journaling). Foster their curiosity about numbers and mathematical ideas. Facilitate research and exploration through discussion, experiments, internet searches, trips, etc. Have students compete in a secondary simultaneous competition, such as skip- counting by 75s, while peers are completing the sprint. Let students choose their mode of response: written, oral, concrete, pictorial, or abstract. Increase the pace. Offer two word problems to solve, rather than one. Adjust difficulty level by increasing the number of steps (e.g., change a one-step problem to a two- step problem). Adjust difficulty level by enhancing the operation (e.

	the meaning of all the words in this problem?; What is being asked?; Do I have all of the information I need?; What do I do first?; What is the order to solve this problem? What calculations do I need to make? Practice routine to ensure smooth transitions. Set goals with students regarding the type of math work students should complete in 60 seconds. Set goals with the students regarding next steps and what to focus on next. Reinforce foundational standards (listed after priority standards) for the unit.	g., addition to multiplication), increasing numbers to millions, or decreasing numbers to decimals/fractions. Let students write word problems to show mastery and/or extension of the content. **<u>Provide Multiple Means of Engagement</u>** Push student comprehension into higher levels of Bloom's Taxonomy with questions such as: "What would happen if...?""Can you propose an alternative...?""How would you evaluate...?""What choice would you have made...?" Ask "Why?" and "What if?" questions. Celebrate improvement in completion time (e.g., Sprint A completed in 45 seconds and Sprint B completed in 30 seconds). Make the most of the fun exercises for practicing skip-counting. Accept and elicit student ideas and suggestions for ways to extend games. Cultivate student persistence in problem-solving and do not neglect their need for guidance and support.

Appendix A: Lesson Plan Sample

Module 1 Lesson 1

The following is a sample lesson plan from EngageNY. The lesson in its entirety can be found at https://www.engageny.org/resource/precalculus-and-advanced-topics-module-1-topic-a-lesson-1

In addition to the lesson plan, printable worksheets, sample student answers are available online. EngageNY can be used as a resource for all modules.

Lesson 1: Wishful Thinking—Does Linearity Hold?

Student Outcomes

Students learn when ideal linearity properties do and do not hold for classes of functions studied in previous years. Students develop familiarity with linearity conditions.

Lesson Notes

This is a two-day lesson in which we introduce a new definition of a linear transformation and look at common mistakes that students make when assuming that all linear functions meet the requirements for this new definition. A linear transformation is not equivalent to a linear function, which is a function whose graph is a line and can be written as □ =□*x* + □. In this sequence of lessons, a linear transformation is defined as it is in linear algebra courses, which is that a function is linear if it satisfies two conditions: □(□ + □) = □(□) + □(□) and □(□*x*) = □(*f*□). This definition leads to surprising results when students study the function □(□) = 3□ + 1. Students apply this new definition of linear transformation to classes of functions learned in previous years and explore why the conditions for linearity sometimes produce false statements. Students then solve to find specific solutions when the conditions for linearity produce true statements, giving the appearance that a linear function is a linear transformation when it is not. In Lesson 1, students explore polynomials and radical equations. Lesson 2 extends this exploration to trigonometric, rational, and logarithmic functions. Lessons 1 and 2 focus on linearity for real-numbered inputs but lead to the discovery of complex solutions and launch the study of complex numbers. This study includes operations on complex numbers as well as the use of conjugates to find moduli and quotients.

Classwork

Exploratory Challenge (13 minutes)

In this Exploratory Challenge, students work individually while discussing the steps as a class. Students complete the exercises in pairs with the class coming together at the end to present their findings and to watch a video.

Wouldn't it be great if functions were sensible and behaved the way we expected them to do?

Let $f(x) = 2x$ and $g(x) = 3x + 1$.

Write down three facts that you know about $f(x)$ and $g(x)$.

Answers will vary. Both graphs are straight lines. $f(x)$ has a y-intercept of 0. $g(x)$ has a y-intercept of 1. The slope of $f(x)$ is 2. The slope of $g(x)$ is 3.

Which of these functions is linear?

Students will probably say both because they are applying a prior definition of a linear function: $y = mx + b$. Introduce the following definition.

A function is a linear transformation if $f(a + b) = f(a) + f(b)$ and $f(kx) = kf(x)$.

Based on this definition, which function is a linear transformation? Explain how you know.

$f(x) = 2x$ is a linear transformation because $2(a + b) = 2a + 2b$ and $2(kx) = k(2x)$.

$g(x) = 3x + 1$ is not a linear transformation because $3(a + b) + 1 \neq (3a + 1) + (3b + 1)$ and $3(kx) + 1 \neq k(3x + 1)$.

Is $h(x) = 2x - 3$ a linear transformation? Explain.

$h(x) = 2x - 3$ is not a linear transformation because $2(a + b) - 3 \neq (2a - 3) + (2b - 3)$ and $2(kx) - 3 \neq k(2x - 3)$.

Is $j(x) = \frac{1}{2} x$ a linear transformation? Explain.

$j(x) = \frac{1}{2} x$ is a linear transformation because $\frac{1}{2}(a + b) = \frac{1}{2} a + \frac{1}{2} b$ and $\frac{1}{2}(kx) = k\left(\frac{1}{2} x\right)$.

Let $m(x) = x^2$.

Is $m(x)$ a linear transformation?

No. $(a + b)^2 \neq a^2 + b^2$, and $(kx)^2 \neq k(x)^2$.

A common mistake made by many math students is saying $(a + b)^2 = a^2 + b^2$. How many of you have made this mistake before?

Does $(a + b)^2 = a^2 + b^2$? Justify your claim.

Substitute some values of a and b into this equation to show that this statement is not generally true.

Answers will vary, but students could choose $a = 1$ and $b = 1$. In this case, $(1 + 1)^2 = 1^2 + 1^2$ leads to $4 = 2$, which we know is not true. There are many other choices.

Did anyone find values of a and b that made this statement true?

Answers will vary but could include □ = 0, □ = 0 or □ = 1, □ = 0 or □ = 0, □ = 1.

We can find all values of □ and □ for which this statement is true by solving for one of the variables. I want half the class to solve this equation for □ and the other half to solve for □.

Expanding the left side and then combining like terms gives

□^{2} + 2□*b* + □^{2} = □^{2} + □^{2}

2□□ = 0.

This leads to □ = 0 if students are solving for □ and □ = 0 if students are solving for □.

We have solutions for two different variables. Can you explain this to your neighbor?

If □ = 0 and/or □ = 0, the statement (□ + □)^2 = □^{2} + □^{2} is true.

Take a moment and discuss with your neighbor what we have just shown. What statement is true for all real values of □ and □?

(□ + □)^2 = □^{2} + □^{2} is true for only certain values of □ and □, namely if either or both variables equal 0. The statement that is true for all real numbers is (□ + □)^2 = □^{2} + 2□*b* + □^{2}.

A function is a linear transformation when the following are true: □(□*x*) = □*f*(□) and □(□ + □) = □(□) + □(□). We call this function a linear transformation.

Repeat what I have just said to your neighbor.

Students repeat.

Look at the functions □(□) = 2□ and □(□) = □^{2} listed above. Which is a linear transformation? Explain.

□(□) = 2□ is a linear transformation because □(□) = □(□) and □(□ + □) = □(□) + □(□).

□(□) = □^{2} is not a linear transformation □(□) ≠ □(□) and □(□ + □) ≠ □(□) + □(□).

We are just introducing linear transformations in Lessons 1 and 2. This will lead to our discussion in Lesson 3 on when functions are linear transformations. In Lesson 3, students discover that a function whose graph is a line may or may not be a linear transformation.

Exercises 1–2 (10 minutes)

In the exercises below, instruct students to work in pairs and to go through the same steps that they went through in the Exploratory Challenge. Call the class back together, and have groups present their results. You can assign all groups both examples or assign half the class Exercise 1 and the other half Exercise 2. Exercise 2 is slightly more difficult than Exercise 1.

Exercises 1–2

Look at these common mistakes that students make, and answer the questions that follow.

1. If □(□) = √□, does □(□ + □) = □(□) + □(□), when □ and □ are not negative?

a. Can we find a counterexample to refute the claim that □(□ + □) = □(□) + □(□) for all nonnegative values of □ and □?

Answers will vary, but students could choose □ = □ and □ = □. In this case, √(□ + □) = √□ + √□, or √□ = □, which we know is not true. There are many other choices.

b. Find some nonnegative values for □ and □ for which the statement, by coincidence, happens to be true.

Answers will vary but could include □ = □, □ = □ or □ = □, □ = □ or □ = □, □ = □.

c. Find all values of □ and □ for which the statement is true. Explain your work and the results.

√(□ + □) = √□ + √□
(√□ + □)$^{□}$ = √□ $^{□}$□ + √□ $^{□}$
□ + □√□*b* + □ = □ + □
□√□ = □
□*b* = □,

which leads to □ = □ if students are solving for □ and □ = □ if students are solving for □. Anytime □ = □ and/or □ = □, then √(□ + □) = √□ + √□, and the equation is true.

d. Why was it necessary for us to consider only nonnegative values of □ and □?

If either variable is negative, then we would be taking the square root of a negative number, which is not a real number, and we are only addressing real-numbered inputs and outputs here.

e. Does □(□) = √□ display ideal linear properties? Explain.

No, because √□ + □ ≠ √□ + √□ for all real values of the variables.

2. If □(□) = □$^{□}$, does □(□ + □) = □(□) + □(□)?

a. Substitute in some values of □ and □ to show this statement is not true in general.

Answers will vary, but students could choose □ = □ and □ = □. In this case, (□ + □)$^{□}$ = □$^{□}$ + □$^{□}$ or □ = □, which we know is not true. There are many other choices.

b. Find some values for □□ and □□ for which the statement, by coincidence, happens to work.

Answers will vary but could include □ = □, □ = □ or □ = □, □ = □ or □ = □, □ = □.

c. Find all values of □ and □ for which the statement is true. Explain your work and the results.

(□ + □)$^{□}$ = □$^{□}$ + □$^{□}$
□$^{□}$ + □□$^{□}$□ + □*a*□$^{□}$ + □$^{□}$ = □$^{□}$ + □$^{□}$
□□$^{□}$□ + □*a*□$^{□}$ = □
□*ab*(□ + □) = □,

which leads to □ = □, □ = □, and □ = −□.

Anytime □ = □ and/or □ = □ or □ = −□, then (□ + □)□ = □□ + □□, and the equation is true.

d. Is this true for all positive and negative values of □ and □? Explain and prove by choosing positive and negative values for the variables.

Yes, since □ = −□ , if □ is positive, the equation would be true if □ was negative. Likewise, if □ is negative, the equation would be true if □ was positive. Answers will vary. If □ = □ and □ = −□, (□ + (−□))$^{□}$ = (□)$^{□}$ + (−□)$^{□}$ meaning □$^{□}$ = □ + (−□) or □ = □. If □ = −□ and □ = □, ((−□) + □)$^{□}$ = (−□)$^{□}$ + (□)$^{□}$ meaning □$^{□}$ = (−□) + □, or □ = □. Therefore, this statement is true for all positive and negative values of □ and □.

e. Does □(□) = □$^{□}$ display ideal linear properties? Explain.

No, because (□ + □)$^{□}$ ≠ □$^{□}$ + □$^{□}$ for all real values of the variables.

Extension Discussion (14 minutes, optional)

As a class, watch this video (7 minutes) that shows another way to justify Exercise 1 (http://www.jamestanton.com/?p=677) . Discuss what groups discovered in the exercises and what was shown in the video. If time allows, let groups present findings and discuss similarities and differences.

Closing (3 minutes)

Ask students to perform a 30-second Quick Write explaining what we learned today using these questions as a guide.

When does (□ + □)^2 = □^{2} + □^{2}? How do you know?

When □ = 0 and/or □ = 0.

When does √□ + □ = √□ + √□? How do you know?

When □ = 0 and/or □ = 0.

Are □ = 0 and/or □ = 0 always the values when functions display ideal linear properties?

No, it depends on the function. Sometimes these values work, and other times they do not. Sometimes there are additional values that work such as with the function $□(□) = □^3$, when $□ = -□$ also works.

When does a function display ideal linear properties?

When $□(□ + □) = □(□) + □(□)$ and $□(□x) = □f(□)$.

Exit Ticket (5 minutes)